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Some qualitative properties of nonlinear fractional integro-differential equations of variable order

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ARTICLE INFO	ABSTRACT
	The existence-uniqueness criteria of nonlinear fractional integro-differential equations of variable order with multiterm boundary value conditions are considered in this work. By utilizing the concepts of generalized intervals combined with the piecewise constant functions, we transform our problem into usual Caputo's fractional differential equations of constant order. We develop the necessary criteria for assuring the solution's existence and uniqueness by applying Schauder and Banach fixed point theorem. We also examine the stability of the derived solution in the Ulam-Hyers-Rassias (UHR) sense and provide an example to demonstrate the credibility of the results.

1. Introduction

The fundamental idea behind fractional calculus is simply to replace the traditional integer orders in integral and differential operators with arbitrary constant orders. Although it seems an elementary consideration, fractional order operators play an important role in describing many physical phenomena and have interesting implications. [1, 2].

The introduction of the notion of variable-order (VO) integral and differential operators together with their some main properties was firstly initialized by Samko and Ross [3] in 1993. By these operators, one can define the order of the fractional integral and derivative as a function of independent variables such as time and space variables. In view of the characterization of the non-fixed kernel, this operators allows us to designate the

memory and hereditary features of natural phenomena in a better way. By virtue of its potential efficiency to model real world problems, this topic has attracted many researchers in ongoing decades. In this direction, lots of papers have been published on different branches of science and engineering such as viscoelasticity, medicine, signal processing, control systems, so on [4–7]. Since its difficulty in getting an explicit solutions for fractional differential equations of VO, many papers have been devoted to find numerical solutions for this type of problems. See [8–13] and the references cited therein. Hovever very few paper on existence, uniqueness and stability properties of fractional variable order differential equations have been published recently [14–20].

When we conduct an overview of the literature, increasing number of authors from several areas of the scientific community have focused on investigating the existence and uniqueness of fractional

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constant order differential and integro-differential equations [21–26].

In [27], Devi et al. studied the following Caputo's fractional boundary value problem by taking into consideration the monotone iterative technique.

$$\begin{cases} {}^C D^q u = F(t, u, I^q(u)) + G(t, u, I^q(u)), \\ g(u(0), u(T)) = 0 \end{cases}$$

where $0 < q < 1$. As a consequence, it has been shown that the established monotone flows converge uniformly to the coupled extremal solutions of the considered problem.

In [28], some sort of stability results were studied for fractional integro-differential equations involving Hilfer fractional derivative ${}^H D_{a^+}^{\alpha, \beta; \psi}(.)$ with $0 < \alpha < 1$ and $0 \le \beta \le 1$.

In particular, Bai and Kong [29] considered the existence of the solutions for the following initial value problem

$$\begin{cases} {}^C D_{a^+}^\alpha y(t) = f(t, y(t), I_{a^+}^\alpha y(t)), & t \in [a, b], \\ y(a) = x_a, \end{cases}$$

by employing the upper and lower solution approach. The operators ${}^C D_{0^+}^\alpha$ and $I_{0^+}^\alpha$ stand for the Caputo-Hadamard fractional derivative and Hadamard fractional integral operators of order $\alpha \in (0, 1]$, respectively.

Motivated by the preceding works, we deal with the following boundary value problem on $\mathcal{J} := [0, b]$ such that

$$\begin{cases} {}^C \mathcal{D}_{0^+}^{u(t)} y(t) = \Phi(t, y(t), I_{0^+}^{u(t)} y(t)), \\ y(0) = 0, \ y(b) = 0, \end{cases} \quad (1)$$

where $1 < u(t) \le 2$ and ${}^C \mathcal{D}_{0^+}^{u(t)}$, $I_{0^+}^{u(t)}$ are considered as in the sense of Caputo fractional derivative and integral of variable-order $u(t)$, respectively.

Our purpose is to investigate the existence and uniqueness of the solution of equation (1). We further show the stability of the solution in the Ulam-Hyers-Rassias (UHR) sense.

2. Mathematical Preliminaries

This part covers some fundamental concepts and lemmas that will be needed to understand the main theorems discussed in the subsequent sections. We also introduce some of the specifications for variable order operators.

Let $C(\mathcal{J}, \mathbb{R})$ denote the the set of all real-valued continuous functions from $\mathcal{J}$ into $\mathbb{R}$. For an element $\chi \in C(\mathcal{J}, \mathbb{R})$, define the standart norm $\|\chi\| = Sup\{|\chi(t)| : t \in \mathcal{J}\}$, and with this norm $C(\mathcal{J}, \mathbb{R})$ becomes a Banach space.

For $-\infty < t_1 < t_2 < +\infty$, let the mappings to be defined $u(t) : [t_1, t_2] \to (0, +\infty)$ and $v(t) :$

$[t_1, t_2] \to (n - 1, n)$. Then, the left Riemann-Lioville(R-L) fractional integral of VO $u(t)$ ([30]) is given as

$$I_{t_1^+}^{u(t)} m(t) = \int_{t_1}^t \frac{(t - s)^{u(t)-1}}{\Gamma(u(t))} m(s) ds, \ t > t_1, \ (2)$$

as well as the left VO Caputo derivative ([30]) is defined by

$${}^C \mathcal{D}_{t_1^+}^{v(t)} m(t) = \int_{t_1}^t \frac{(t - s)^{n-v(t)-1}}{\Gamma(n - v(t))} m^{(n)}(s) ds, \ t > t_1. \quad (3)$$

These definitions, as expected, correspond with the usual R-L fractional integral and Caputo fractional derivative, respectively, when $u(t)$ and $v(t)$ are constant. see e.g. [3, 30, 31].

Lemma 1. *([31]) Let σ_1, $\sigma_2 > 0$, $t_1 > 0$, $m \in L(t_1, t_2)$ and ${}^C \mathcal{D}_{t_1^+} m \in L(t_1, t_2)$. Then, differential equation*

$${}^C \mathcal{D}_{t_1^+}^{\sigma_1} m(t) = 0$$

has the following general solution

$$m(t) = \alpha_0 + \alpha_1(t - t_1) + \alpha_2(t - t_1)^2 + ... + \alpha_{n-1}(t - t_1)^{n-1}$$

where $n - 1 < \sigma_1 \le n$ and α_ℓ ($\ell = 0, 1, ..., n - 1$) are taken as arbitrary real numbers.

From that Lemma we deduce the next relation

$$I_{t_1^+}^{\sigma_1} {}^C \mathcal{D}_{t_1^+}^{\sigma_1} m(t) = m(t) + \alpha_0 + \alpha_1(t - t_1)$$
$$+ \alpha_2(t - t_1)^2 + ... + \alpha_{n-1}(t - t_1)^{n-1}$$

Furthermore,

$${}^C \mathcal{D}_{t_1^+}^{\sigma_1} I_{t_1^+}^{\sigma_1} m(t) = m(t).$$

and

$$I_{t_1^+}^{\sigma_1} I_{t_1^+}^{\sigma_2} m(t) = I_{t_1^+}^{\sigma_2} I_{t_1^+}^{\sigma_1} m(t) = I_{t_1^+}^{\sigma_1 + \sigma_2} m(t).$$

Remark 1. *([32]) It's worth noting that the semigroup property isn't mostly satisfied by general functions $u(t)$, $v(t)$, i.e.,*

$$I_{t_1^+}^{u(t)} I_{t_1^+}^{v(t)} m(t) \ne I_{t_1^+}^{u(t)+v(t)} m(t).$$

Definition 1. *A function $\mu \in C(\mathcal{J}, \mathbb{R})$ is said to be a C_δ class function if it belong to the set*

$$C_\delta(\mathcal{J}, \mathbb{R}) = \left\{ \mu \in C((0, b], \mathbb{R}) : t^\delta \mu \in C(\mathcal{J}, \mathbb{R}) \right\}$$

for $0 \le \delta \le 1$.

Lemma 2. *[13] Assume that $u : \mathcal{J} \to (1, 2)$ is a continuous function and $m \in C_\delta(\mathcal{J}, \mathbb{R})$. Then the fractional integral $I_{0^+}^{u(t)} m(t)$ of variable order exists for each point on $\mathcal{J}$.*

Lemma 3. *([13]) Let $u \in C(\mathcal{J}, (1, 2))$ and $m \in C(\mathcal{J}, \mathbb{R})$ then $I_{0^+}^{u(t)} m(t) \in C(\mathcal{J}, \mathbb{R})$.*

We now give the well-known Schauder fixed-point result.

Theorem 1. *([31]) Assume that E is a Banach space and Q is a nonempty convex subset of E and moreover $F : Q \longrightarrow Q$ is compact, and continuous map. Then, there exist fixed points of F in Q.*

Definition 2. *([23]) The equation (1) is called Ulam-Hyers-Rassias **(UHR)** stable with respect to the function $\psi \in C(\mathcal{J}, \mathbb{R}^+)$ if there exists $c_\Phi > 0$, such that for any $\epsilon > 0$ and for each solution $z \in C^1(\mathcal{J}, \mathbb{R})$ of the inequality*

$$|{}^C\mathcal{D}_{0^+}^{u(t)} z(t) - \Phi(t, z(t), I_{0^+}^{u(t)} z(t))| \leq \epsilon\psi(t), \ t \in \mathcal{J},$$

there exists a solution $y \in C(\mathcal{J}, \mathbb{R})$ of equation (1) with

$$|z(t) - y(t)| \leq c_\Phi \epsilon \psi(t), \ t \in \mathcal{J}.$$

3. Existence Results

Let us begin with introducing the following assumptions:

(H1): Let $\mathcal{P} = \{\mathcal{J}_1 := [0, b_1], \mathcal{J}_2 := (b_1, b_2], \mathcal{J}_3 := (b_2, b_3], ... \mathcal{J}_n := (b_{n-1}, b]\}$ be a partition of the interval $\mathcal{J}$, and let $u(t) : \mathcal{J} \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

$$u(t) = \sum_{\ell=1}^{n} u_\ell I_\ell(t) = \begin{cases} u_1, & if \ t \in \mathcal{J}_1, \\ u_2, & if \ t \in \mathcal{J}_2, \\ . \\ . \\ . \\ u_n, & if \ t \in \mathcal{J}_n, \end{cases}$$

where $1 < u_\ell \leq 2$ are constants, and I_ℓ is the indicator of the interval $\mathcal{J}_\ell := (b_{\ell-1}, b_\ell]$, $\ell = 1, 2, ..., n$, (with $b_0 = 0$, $b_n = b$) such that

$$I_\ell(t) = \begin{cases} 1, & for \ t \in \mathcal{J}_\ell, \\ 0, & for \ elsewhere. \end{cases}$$

(H2): Let $t^\delta \Phi : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function $(0 \leq \delta \leq 1)$, there exist constants $K, \ L > 0$, satisfying the inequality

$$t^\delta |\Phi(t, w_1, z_1) - \Phi(t, w_2, z_2)| \ \leq K|w_1 - w_2| \\ +L|z_1 - z_2|,$$

For each $\ell \in \{1, 2, ..., n\}$, the set $E_\ell = C(\mathcal{J}_\ell, \mathbb{R})$, represents the Banach space of continuous functions $y : \mathcal{J}_\ell \to \mathbb{R}$ equipped with the sup norm

$$\|y\|_{E_\ell} = \sup_{t \in \mathcal{J}_\ell} |y(t)|,$$

where $\ell \in \{1, 2, ..., n\}$
We now analyze BVP defined in (1). On account of (3), the solution of (1) can be stated as

$$\int_0^t \frac{(t-s)^{1-u(t)}}{\Gamma(2 - u(t))} y''(s) ds = \Phi(t, y(t), I_{0^+}^{u(t)} y(t)), \ (4)$$

for $t \in \mathcal{J}$. If we employ $(H1)$, the foregoing equation(4) can be written as

$$\int_0^{b_1} \frac{(t-s)^{1-u_1}}{\Gamma(2 - u_1)} y''(s) ds + ... \\ + \int_{b_{\ell-1}}^t \frac{(t-s)^{1-u_\ell}}{\Gamma(2 - u_\ell)} y''(s) ds = \Phi(t, y(t), I_{0^+}^{u_\ell} y(t)) \tag{5}$$

for $t \in \mathcal{J}_\ell, \ell = 1, 2, ..., n$.
The solution to the BVP (1) will be introduced in the following definition.

Definition 3. *BVP (1) has a solution, if there are functions $y_\ell, \ell = 1, 2, ..., n$, such that $y_\ell \in C([0, b_\ell], \mathbb{R})$ satisfying equation (5) and boundary conditions $y_\ell(0) = 0 = y_\ell(b_\ell)$.*

Based on the preceding observation, BVP (1) can be represented as in (4) and, with considering $\mathcal{J}_\ell, \ell \in \{1, 2, ..., n\}$ as in (5).
Since we define $y(t)$ identically 0 for $t \in [0, b_{\ell-1})$, then the equation (5) is expressed as

$${}^C\mathcal{D}_{b_{\ell-1}^+}^{u_\ell} y(t) = \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)), \quad t \in \mathcal{J}_\ell.$$

We shall deal with following BVP

$$\begin{cases} {}^C\mathcal{D}_{b_{\ell-1}^+}^{u_\ell} y(t) = \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)), \\ y(b_{\ell-1}) = 0, y(b_\ell) = 0, \end{cases} \tag{6}$$

for $t \in \mathcal{J}_\ell$. On the way to achieve our purpose, the upcoming lemma will play an important role.

Lemma 4. *A function $y \in E_\ell$ establishes a solution for (6) if and only if y fulfills the integral equation*

$$y(t) = -\frac{t - b_{\ell-1}}{b_\ell - b_{\ell-1}} \left[I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)) \right]_{t=b_\ell} \\ + I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)). \tag{7}$$

Proof. We first assume that $y \in E_\ell$ is solution of the problem (6). If we apply the fractional operator $I_{b_{\ell-1}^+}^{u_\ell}$ to both sides of (6) and considering Lemma 1, we obtain

$$y(t) = \alpha_1 + \alpha_2(t - b_{\ell-1}) + \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^t (t-s)^{u_\ell-1} \\ \times \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds,$$

for $t \in \mathcal{J}_\ell$. By $y(b_{\ell-1}) = 0$, we get $\alpha_1 = 0$.
Taking into account another boundary condition

$y(b_\ell) = 0$, it follows that

$$
\begin{aligned}
0 \;=\;& \alpha_2(b_\ell - b_{\ell-1}) \\
&+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\
&\times \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds
\end{aligned}
$$

$$
\alpha_2 \;=\; -(b_\ell - b_{\ell-1})^{-1} I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)) \Big|_{t=b_\ell}
$$

Then, we observe that

$$
\begin{aligned}
y(t) \;=\;& -(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1}) \\
&\times \left[I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)) \right]_{t=b_\ell} \\
&+ I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)), \quad t \in \mathcal{J}_\ell.
\end{aligned}
$$

On the contrary, let $y \in E_\ell$ be the solution of integral equation (7). Taking into account the continuity of function $t^\delta \Phi$ and using Lemma (1), we conclude that y is the solution of the problem (6). $\square$

We can now show our first existence result which is based on Theorem (1)

Theorem 2. *Assume that conditions (H1), (H2) hold, and if*

$$
\frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\left(K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right) \;<\; 1,
$$

then, there exist at least one solution for the problem (6) on $\mathcal{J}$.

Proof. Let us set the operator $W : E_\ell \to E_\ell$ such that for $t \in \mathcal{J}_\ell$

$$
\begin{aligned}
Wy(t) \;=\;& -(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1}) \\
&\times \left[I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)) \right]\Big|_{t=b_\ell} \\
&+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1} \\
&\times \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds.
\end{aligned}
$$

The operator $W : E_\ell \to E_\ell$ described in 3.1 is well defined, as seen by the properties of fractional integrals and the continuity of function $t^\delta \Phi$.
Let

$$
R_\ell \;\geq\; \frac{\frac{2\eta_0(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)}}{1 - \frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\left(K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)},
$$

with

$$
\eta_0 = \sup_{t \in \mathcal{J}_\ell} |\Phi(t, 0, 0)|.
$$

We generate the set

$$
B_{R_\ell} = \{y \in E_\ell : \; \|y\|_{E_\ell} \leq R_\ell\}.
$$

It is clear that B_{R_ℓ} is nonempty, closed, convex and bounded.

Now, we will see that W satisfies the claims of the Theorem (1). We demonstrate it by using following stages.

STEP 1: We show that $W(B_{R_\ell}) \subseteq (B_{R_\ell})$.
For $y \in B_{R_\ell}$ and by (H2), we get

$$
\begin{aligned}
|Wy(t)| \;\leq\;& \frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \\
&\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\
&\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds \\
&+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1} \\
&\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds \\
\leq\;& \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\
&\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds \\
=\;& \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\
&\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) - \Phi(s,0,0)| ds \\
&+ \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} |\Phi(s,0,0)| ds \\
\leq\;& \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} s^{-\delta} \\
&\times (K|y(s)| + L|I_{b_{\ell-1}^+}^{u_\ell} y(s)|) ds \\
&+ \frac{2\eta_0(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
\leq\;& \frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} \\
&\times (K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})|y(s)| ds \\
&+ \frac{2\eta_0(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
\leq\;& \frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \\
&\times (K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})R_\ell \\
&+ \frac{2\eta_0(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
\leq\;& R_\ell,
\end{aligned}
$$

which yields that $W(B_{R_\ell}) \subseteq B_{R_\ell}$.
STEP 2: W is continuous.
We assume that the sequence (y_n) converges to y

in E_ℓ. Then,

$|(Wy_n)(t) - (Wy)(t)|$

$\leq \frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1}$

$\times |\Phi(s, y_n(s), I_{b_{\ell-1}^+}^{u_\ell} y_n(s)) - \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1}$

$\times |\Phi(s, y_n(s), I_{b_{\ell-1}^+}^{u_\ell} y_n(s)) - \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds$

$\leq \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1}$

$\times |\Phi(s, y_n(s), I_{b_{\ell-1}^+}^{u_\ell} y_n(s)) - \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds$

$\leq \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1}$

$\times (K|y_n(s) - y(s)| + L I_{b_{\ell-1}^+}^{u_\ell} |y_n(s) - y(s))|) ds$

$\leq \frac{2K}{\Gamma(u_\ell)} \|y_n - y\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds$

$+ \frac{2L}{\Gamma(u_\ell)} \|I_{b_{\ell-1}^+}^{u_\ell} (y_n - y)\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds$

$\leq \frac{2K}{\Gamma(u_\ell)} \|y_n - y\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds$

$+ \frac{2L(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell + 1)} \|y_n - y\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds$

$\leq \left(\frac{2K}{\Gamma(u_\ell)} + \frac{2L(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell + 1)} \right) \|y_n - y\|_{E_\ell}$

$\times \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds$

$\leq \frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$

$\times \left(K + L \frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \right) \|y_n - y\|_{E_\ell}$ i.e., we obtain

$$\|(Wy_n) - (Wy)\|_{E_\ell} \to 0 \quad as \quad n \to \infty.$$

As a consequence, the operator W is a continuous on E_ℓ.

STEP 3: W is compact.

We will prove that $W(B_{R_\ell})$ is relatively compact, which means that W is compact. In view of step 1, $W(B_{R_\ell})$ is uniformly bounded. Namely, we have $W(B_{R_\ell}) = \{W(y) : y \in B_{R_\ell}\} \subset B_{R_\ell}$ thus for each $y \in B_{R_\ell}$ we get $\|W(y)\|_{E_\ell} \leq R_\ell$ showing that $W(B_{R_\ell})$ is bounded. Finally, It must be demonstrated the equicontinuity of $W(B_{R_\ell})$. For $t_1, t_2 \in \mathcal{J}_\ell$, $t_1 < t_2$ and $y \in B_{R_\ell}$, we write

$|(Wy)(t_2) - (Wy)(t_1)|$

$= \left| - \frac{(b_\ell - b_{\ell-1})^{-1}(t_2 - b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \right.$

$\times \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t_2} (t_2 - s)^{u_\ell - 1} \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds$

$+ \frac{(b_\ell - b_{\ell-1})^{-1}(t_1 - b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1}$

$\times \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds$

$\left. - \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t_1} (t_1 - s)^{u_\ell - 1} \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds \right|$

$\leq \frac{(b_\ell - b_{\ell-1})^{-1}}{\Gamma(u_\ell)} \left((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \right)$

$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t_1} \left((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1} \right)$

$\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{t_1}^{t_2} (t_2 - s)^{u_\ell - 1} |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds$

$\leq \frac{(b_\ell - b_{\ell-1})^{-1}}{\Gamma(u_\ell)} \left((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \right)$

$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \left| \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) \right.$

$\left. - \Phi(s, 0, 0) \right| ds$

$+ \frac{(b_\ell - b_{\ell-1})^{-1}}{\Gamma(u_\ell)} \left((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \right)$

$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} |\Phi(s, 0, 0)| ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t_1} \left((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1} \right)$

$\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) - \Phi(s, 0, 0)| ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t_1} \left((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1} \right)$

$\times |\Phi(s, 0, 0)| ds + \frac{1}{\Gamma(u_\ell)} \int_{t_1}^{t_2} (t_2 - s)^{u_\ell - 1}$

$\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) - \Phi(s, 0, 0)| ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{t_1}^{t_2} (t_2 - s)^{u_\ell - 1} |\Phi(s, 0, 0)| ds$

$\leq \frac{(b_\ell - b_{\ell-1})^{-1}}{\Gamma(u_\ell)} \left((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \right)$

$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \left| \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) \right.$

$\left. - \Phi(s, 0, 0) \right| ds$

$+ \frac{(b_\ell - b_{\ell-1})^{-1}}{\Gamma(u_\ell)} \left((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \right)$

$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} |\Phi(s, 0, 0)| ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t_1} \left((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1} \right)$

$\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) - \Phi(s, 0, 0)| ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t_1} \left((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1} \right)$

$\times |\Phi(s, 0, 0)| ds + \frac{1}{\Gamma(u_\ell)} \int_{t_1}^{t_2} (t_2 - s)^{u_\ell - 1}$

$\times \, |\Phi(s, y(s), I^{u_\ell}_{b^+_{\ell-1}} y(s)) - \Phi(s, 0, 0)|ds$

$+ \frac{1}{\Gamma(u_\ell)} \int^{t_2}_{t_1} (t_2 - s)^{u_\ell - 1} |\Phi(s, 0, 0)|ds.$

$\leq \frac{(b_\ell - b_{\ell-1})^{-1}}{\Gamma(u_\ell)} ((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}))$

$\times \int^{b_\ell}_{b_{\ell-1}} (b_\ell - s)^{u_\ell - 1} s^{-\delta} (K|y(s)| + L|I^{u_\ell}_{b^+_{\ell-1}} y(s)|)ds$

$+ \frac{\eta_0 (b_\ell - b_{\ell-1})^{-1}}{\Gamma(u_\ell)} \Big((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \Big)$

$\times \int^{b_\ell}_{b_{\ell-1}} (b_\ell - s)^{u_\ell - 1} ds$

$+ \frac{1}{\Gamma(u_\ell)} \int^{t_1}_{b_{\ell-1}} s^{-\delta} \Big((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1} \Big)$

$\times (K|y(s)| + L|I^{u_\ell}_{b^+_{\ell-1}} y(s)|)ds$

$+ \frac{\eta_0}{\Gamma(u_\ell)} \int^{t_1}_{b_{\ell-1}} \Big((t_2 - s)^{u_\ell - 1} - (t_1 - s)^{u_\ell - 1} \Big) ds$

$+ \frac{1}{\Gamma(u_\ell)} \int^{t_2}_{t_1} s^{-\delta} (t_2 - s)^{u_\ell - 1}$

$\times (K|y(s)| + L|I^{u_\ell}_{b^+_{\ell-1}} y(s)|)ds$

$+ \frac{\eta_0}{\Gamma(u_\ell)} \int^{t_2}_{t_1} (t_2 - s)^{u_\ell - 1} ds$

$\leq \frac{(b_\ell - b_{\ell-1})^{u_\ell - 2}}{\Gamma(u_\ell)} \Big((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \Big)$

$\times (K\|y\|_{E_\ell} + L\|I^{u_\ell}_{b^+_{\ell-1}} y\|_{E_\ell}) \int^{b_\ell}_{b_{\ell-1}} s^{-\delta} ds$

$+ \frac{\eta_0 (b_\ell - b_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell + 1)} \Big((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \Big)$

$+ \frac{1}{\Gamma(u_\ell)} (K\|y\|_{E_\ell} + L\|I^{u_\ell}_{b^+_{\ell-1}} y\|_{E_\ell})$

$\times \int^{t_1}_{b_{\ell-1}} s^{-\delta} ((t_2 - t_1)^{u_\ell - 1}) ds$

$+ \frac{\eta_0}{\Gamma(u_\ell)} \Big(\frac{(t_2 - b_{\ell-1})^{u_\ell}}{u_\ell} - \frac{(t_2 - t_1)^{u_\ell}}{u_\ell} - \frac{(t_1 - b_{\ell-1})^{u_\ell}}{u_\ell} \Big)$

$+ \frac{(t_2 - t_1)^{u_\ell - 1}}{\Gamma(u_\ell)} (K\|y\|_{E_\ell} + L\|I^{u_\ell}_{b^+_{\ell-1}} y\|_{E_\ell}) \int^{t_2}_{t_1} s^{-\delta} ds$

$+ \frac{\eta_0}{\Gamma(u_\ell)} \frac{(t_2 - t_1)^{u_\ell}}{u_\ell}$

$\leq \frac{(b_\ell - b_{\ell-1})^{u_\ell - 2} (b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$

$\times ((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}))$

$\times (K\|y\|_{E_\ell} + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \|y\|_{E_\ell})$

$+ \frac{\eta_0 (b_\ell - b_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell + 1)} \Big((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \Big)$

$+ \Big(\frac{(t_1^{1-\delta} - b_{\ell-1}^{1-\delta})(t_2 - t_1)^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)} \Big)$

$\times (K\|y\|_{E_\ell} + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \|y\|_{E_\ell}) + \frac{\eta_0}{\Gamma(u_\ell + 1)}$

$\times \Big((t_2 - b_{\ell-1})^{u_\ell} - (t_2 - t_1)^{u_\ell} - (t_1 - b_{\ell-1})^{u_\ell} \Big)$

$+ \frac{(t_2^{1-\delta} - t_1^{1-\delta})(t_2 - t_1)^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)}$

$\times (K\|y\|_{E_\ell} + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \|y\|_{E_\ell}) + \frac{\eta_0 (t_2 - t_1)^{u_\ell}}{\Gamma(u_\ell + 1)}$

$\leq \Big(\frac{(b_\ell - b_{\ell-1})^{u_\ell - 2}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} (K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)})$

$\times \|y\|_{E_\ell} + \frac{\eta_0 (b_\ell - b_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell + 1)} \Big)$

$\times \Big((t_2 - b_{\ell-1}) - (t_1 - b_{\ell-1}) \Big)$

$+ \Big(\frac{t_2^{1-\delta} - b_{\ell-1}^{1-\delta}}{(1-\delta)\Gamma(u_\ell)} (K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}) \|y\|_{E_\ell} \Big)$

$\times (t_2 - t_1)^{u_\ell - 1}$

$+ \frac{\eta_0}{\Gamma(u_\ell + 1)} \Big((t_2 - b_{\ell-1})^{u_\ell} - (t_1 - b_{\ell-1})^{u_\ell} \Big)$

Hence $\|(Wy)(t_2) - (Wy)(t_1)\|_{E_\ell} \to 0$ as $|t_2 - t_1| \to 0$. It implies that $W(B_{R_\ell})$ is equicontinuous.

As a consequence of the Theorem (1), the problem (6) has at least a solution $\widetilde{y}_\ell$ in B_{R_ℓ}. Let

$$y_\ell = \begin{cases} 0, & t \in [0, b_{\ell-1}], \\ \\ \widetilde{y}_\ell, & t \in \mathcal{J}_\ell, \end{cases} \tag{8}$$

We know that $y_\ell \in C([0, b_\ell], X)$ defined by (8) satisfies the equation

$$\int^{b_1}_0 \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} y_\ell''(s)ds + \dots$$

$$+ \int^t_{b_{\ell-1}} \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} y_\ell''(s)ds = \Phi(t, y_\ell(t), I^{u_\ell}_{0^+} y_\ell(t)),$$

for $t \in \mathcal{J}_\ell$, concluding that y_ℓ is a solution of (5) with $y_\ell(0) = 0$, $y_\ell(b_\ell) = \widetilde{y}_\ell(b_\ell) = 0$.

Then,

$$y(t) = \begin{cases} y_1(t), & t \in \mathcal{J}_1, \\ y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1, \\ \widetilde{y}_2, & t \in \mathcal{J}_2 \end{cases} \\ . \\ . \\ . \\ y_n(t) = \begin{cases} 0, & t \in [0, b_{\ell-1}], \\ \widetilde{y}_\ell, & t \in \mathcal{J}_\ell \end{cases} \end{cases} .$$

constitutes a solution for BVP(1).

The principle of Banach contraction is used to arrive at the following result. $\qquad \square$

Theorem 3. *Assume that the assumptions $(H1)$, $(H2)$ hold and if*

$$\frac{2(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})(b_\ell - b_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)}$$

$$\times \Big(K + \frac{L(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \Big) < 1 \tag{9}$$

then the problem (6) has at most one solution in E_ℓ.

Proof. The Banach contraction concept will be used to demonstrate the unique fixed point for W specified in Theorem (3).

For $y_1(t), y_2(t) \in E_\ell$, it follows that

$$|(Wy_1)(t) - (Wy_2)(t)|$$

$$\leq \frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1}$$

$$\times |\Phi(s, y_1(s), I_{b_{\ell-1}^+}^{u_\ell} y_1(s)) - \Phi(s, y_2(s), I_{b_{\ell-1}^+}^{u_\ell} y_2(s))| ds$$

$$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1}$$

$$\times |\Phi(s, y_1(s), I_{b_{\ell-1}^+}^{u_\ell} y_1(s)) - \Phi(s, y_2(s), I_{b_{\ell-1}^+}^{u_\ell} y_2(s))| ds$$

$$\leq \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1}$$

$$\times |\Phi(s, y_1(s), I_{b_{\ell-1}^+}^{u_\ell} y_1(s)) - \Phi(s, y_2(s), I_{b_{\ell-1}^+}^{u_\ell} y_2(s))| ds$$

$$\leq \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} s^{-\delta}$$

$$\times \left(K|y_1(s) - y_2(s)| + L I_{b_{\ell-1}^+}^{u_\ell} |y_1(s) - y_2(s))| \right) ds$$

$$\leq \frac{2K}{\Gamma(u_\ell)} \|y_1 - y_2\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta}(b_\ell - s)^{u_\ell - 1} ds$$

$$+ \frac{2L}{\Gamma(u_\ell)} \|I_{b_{\ell-1}^+}^{u_\ell}(y_1 - y_2)\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta}(b_\ell - s)^{u_\ell - 1} ds$$

$$\leq \frac{2K}{\Gamma(u_\ell)} \|y_1 - y_2\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta}(b_\ell - s)^{u_\ell - 1} ds$$

$$+ \frac{2L(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)} \|y_1 - y_2\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta}(b_\ell - s)^{u_\ell - 1} ds$$

$$\leq \left(\frac{2K}{\Gamma(u_\ell)} + \frac{2L(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)} \right) \|y_1 - y_2\|_{E_\ell}$$

$$\times \int_{b_{\ell-1}}^{b_\ell} s^{-\delta}(b_\ell - s)^{u_\ell - 1} ds$$

$$\leq \frac{2(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})(b_\ell - b_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)}$$

$$\times \left(K + \frac{L(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} \right) \|y_1 - y_2\|_{E_\ell}$$

Therefore, by considering (9), the operator W is a contraction. Employing Banach contraction mapping, we result in that W has only one fixed point, say it $\widetilde{y}_\ell \in E_\ell$, which also concludes that the problem (6) has got unique solution. We let

$$y_\ell = \begin{cases} 0, & t \in [0, b_{\ell-1}], \\ \\ \widetilde{y}_\ell, & t \in \mathcal{J}_\ell, \end{cases} \tag{10}$$

We know that $y_\ell \in C([0, b_\ell], \mathbb{R})$ defined by (10) satisfies the equation

$$\int_0^{b_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} y_\ell''(s) ds + \ldots$$

$$+ \int_{b_{\ell-1}}^{t} \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} y_\ell''(s) ds = \Phi(t, y_\ell(t), I_{0^+}^{u_\ell} y_\ell(t)),$$

for $t \in \mathcal{J}_\ell$, which yields that y_ℓ is a unique solution of (5) with $y_\ell(0) = 0$, $y_\ell(b_\ell) = \widetilde{y}_\ell(b_\ell) = 0$.

This led us to

$$y(t) = \begin{cases} y_1(t), & t \in \mathcal{J}_1, \\ y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1, \\ \widetilde{y}_2, & t \in \mathcal{J}_2 \end{cases} \\ \vdots \\ \vdots \\ y_n(t) = \begin{cases} 0, & t \in [0, b_{\ell-1}], \\ \widetilde{x}_\ell, & t \in \mathcal{J}_\ell \end{cases} \end{cases} .$$

which is the unique solution of the boundary value problem (1). $\qquad\square$

4. Ulam-Hyers-Rassias stability

Theorem 4. *Suppose that the conditions (H1), (H2), together with (9) hold. Assume further that*

(H3): *The function $\psi \in C(\mathcal{J}_\ell, \mathbb{R}_+)$ have increasing property and there exists $\lambda_\psi > 0$ such that*

$$I_{b_{\ell-1}^+}^{u_\ell} \psi(t) \leq \lambda_\psi \psi(t)$$

then, under these assumptions, the equation (1) has **UHR** *stability with respect to ψ*

Proof. Suppose that $z \in C(\mathcal{J}_\ell, \mathbb{R})$ is a solution of the following inequality

$$\left|{}^C\mathcal{D}_{b_{\ell-1}^+}^{u_\ell} z(t) - \Phi(t, z(t), I_{b_{\ell-1}^+}^{u_\ell} z(t))\right| \leq \epsilon\psi(t), \tag{11}$$

for $t \in \mathcal{J}_\ell$. Let us denote $y \in C(\mathcal{J}_\ell, \mathbb{R})$ to be the unique solution of the problem

$$\begin{cases} {}^C\mathcal{D}_{b_{\ell-1}^+}^{u_\ell} y(t) = \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)), & t \in \mathcal{J}_\ell \\ y(b_{\ell-1}) = 0, & y(b_\ell) = 0 \end{cases}$$

By using Lemma (4), we have

$$y(t) = -\frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1}$$

$$\times \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds$$

$$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1}$$

$$\times \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds$$

By integrating both sides of (11) and utilizing (H3), we find

$$\left| z(t) + \frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \right.$$

$$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z(s)) ds$$

$$\left. - \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1} \Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z(s)) ds \right|$$

$$\leq \epsilon \int_{b_{\ell-1}}^{t} \frac{(t-s)^{u_\ell - 1}}{\Gamma(u_\ell)} \psi(s) ds$$

$$\leq \epsilon \lambda_\psi \psi(t)$$

In addition, we get for each $t \in \mathscr{J}_\ell$

$$|z(t) - y(t)|$$

$$= \left| z(t) + \frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \right.$$

$$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds$$

$$\left. - \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1} \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds \right|$$

$$\leq \left| z(t) + \frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \right.$$

$$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z(s)) ds$$

$$\left. - \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1} \Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z(s)) ds \right|$$

$$+ \frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1}$$

$$\times |\Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z) - \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y)| ds$$

$$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1}$$

$$\times |\Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z) - \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y)| ds$$

$$\leq \lambda_\psi \epsilon \psi(t)$$

$$+ \frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} s^{-\delta}$$

$$\times (K|z(s) - y(s)| + L I_{b_{\ell-1}^+}^{u_\ell} |z(s) - y(s)|) ds$$

$$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1} s^{-\delta}$$

$$\times (K|z(s) - y(s)| + L I_{b_{\ell-1}^+}^{u_\ell} |z(s) - y(s)|) ds$$

$$\leq \lambda_\psi \epsilon \psi(t) + \frac{(b_\ell - b_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)})$$

$$\times (K\|z - y\|_{E_\ell} + L\|I_{b_{\ell-1}^+}^{u_\ell}(z - y)\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} ds$$

$$+ \frac{(b_\ell - b_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)}(K\|z - y\|_{E_\ell} + L\|I_{b_{\ell-1}^+}^{u_\ell}(z - y)\|_{E_\ell})$$

$$\times \int_{b_{\ell-1}}^{t} s^{-\delta} ds$$

$$\leq \lambda_\psi \epsilon \psi(t) + \frac{(b_\ell - b_{\ell-1})^{u_\ell - 1}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$$

$$\times (K\|z - y\|_{E_\ell} + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\|z - y\|_{E_\ell})$$

$$+ \frac{(b_\ell - b_{\ell-1})^{u_\ell - 1}(t^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$$

$$\times (K\|z - y\|_{E_\ell} + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\|z - y\|_{E_\ell})$$

$$\leq \lambda_\psi \epsilon \psi(t) + \frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$$

$$\times (K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)})\|z - y\|_{E_\ell}$$

which gives

$$\|z - y\|_{E_\ell}$$

$$\times \left(1 - \frac{2(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})(b_\ell - b_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)} \right.$$
$$\left. \times (K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}) \right)$$

$$\leq \lambda_\psi \epsilon \psi(t)$$

For each $t \in \mathscr{J}_\ell$, we arrive at the following relation

$$\|z - y\|_{E_\ell}$$

$$\leq \frac{\lambda_\psi \epsilon \psi(t)}{(1 - \frac{2(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})(b_\ell - b_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)}(K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}))}$$

$$= [1 - \frac{2(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})(b_\ell - b_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)}$$

$$\times (K + L\frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)})]^{-1} \lambda_\psi \epsilon \psi(t)$$

$$:= c_\Phi \epsilon \psi(t)$$

which concludes that the equation (6) admits **UHR** stability with respect to ψ for each $i \in \{1, 2, ..., n\}$.

Consequently, main problem (1) has **UHR** stability with respect to ψ. $\qquad\square$

5. Example

Consider the fractional boundary value problem that follows:

$$\begin{cases} {}^C\mathcal{D}_{0+}^{u(t)}y(t) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{u(t)}y(t)|)}, \\[12pt] y(0) = 0, \;\; y(2) = 0. \end{cases}$$

$$(12)$$

for $t \in \mathcal{J} := [0,2]$,
Let

$$\Phi(t,y,z) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}} + 4e^{2t} + 1)(1+y+z)},$$

$(t,y,z) \in [0,2] \times [0,+\infty) \times [0,+\infty)$ and

$$u(t) = \begin{cases} \dfrac{3}{2}, & t \in \mathcal{J}_1 := [0,1], \\[10pt] \dfrac{9}{5}, & t \in \mathcal{J}_2 :=]1,2]. \end{cases} \qquad (13)$$

Then, we have

$$t^{\frac{1}{3}}|\Phi(t,w_1,z_1) - \Phi(t,w_2,z_2)|$$

$$= \left| \dfrac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)} \left(\dfrac{1}{1+w_1+z_1} - \dfrac{1}{1+w_2+z_2} \right) \right|$$

$$\leq \dfrac{e^{-t}(|w_1-w_2|+|z_1-z_2|)}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+w_1+z_1)(1+w_2+z_2)}$$

$$\leq \dfrac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)}(|w_1-w_2|+|z_1-z_2|)$$

$$\leq \dfrac{1}{(e+5)}|w_1-w_2| + \dfrac{1}{(e+5)}|z_1-z_2|.$$

As a result, with $\delta = \frac{1}{3}$ and $K = L = \frac{1}{e+5}$, the assumption (H2) is satisfied.
By (13), solution of the given problem (12) can be split into two parts as follows

$$\begin{cases} {}^C\mathcal{D}_{0+}^{\frac{3}{2}}y(t) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{\frac{3}{2}}y(t)|)}, \\[8pt] t \in \mathcal{J}_1, \\[6pt] {}^C\mathcal{D}_{1+}^{\frac{9}{5}}y(t) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{\frac{9}{5}}y(t)|)}, \\[8pt] t \in \mathcal{J}_2. \end{cases}$$

For $t \in \mathcal{J}_1$, we begin by looking at the following boundary value problem:

$$\begin{cases} {}^C\mathcal{D}_{0+}^{\frac{3}{2}}y(t) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{\frac{3}{2}}y(t)|)}, \\[8pt] y(0) = 0, \;\; y(1) = 0. \end{cases}$$

$$(14)$$

We are in position to check whether the condition (9) is satisfied or not

$$\dfrac{(b_1^{1-\delta}-b_0^{1-\delta})(b_1-b_0)^{u_1-1}}{(1-\delta)\Gamma(u_1)} \left(2K + \dfrac{2L(b_1-b_0)^{u_1}}{\Gamma(u_1+1)} \right)$$

$$= \dfrac{2}{\frac{2}{3}(e+5)\Gamma(\frac{3}{2})} \left(1 + \dfrac{1}{\Gamma(\frac{5}{2})} \right) \simeq 0.7685 < 1$$

Let $\psi(t) = t^{\frac{1}{2}}$

$$\begin{aligned} I_{0+}^{u_1}\psi(t) &= \dfrac{1}{\Gamma(\frac{3}{2})} \int_0^t (t-s)^{\frac{1}{2}}s^{\frac{1}{2}}ds \\[8pt] &\leq \dfrac{1}{\Gamma(\frac{3}{2})} \int_0^t (t-s)^{\frac{1}{2}}ds \\[8pt] &\leq \dfrac{2}{3\Gamma(\frac{3}{2})}\psi(t) := \lambda_\psi \psi(t). \end{aligned}$$

It shows that the assumption (H3) holds with $\psi(t) = t^{\frac{1}{2}}$ and $\lambda_\psi = \frac{2}{3\Gamma(\frac{3}{2})}$.
Regarding Theorem (3), the problem (14) has a unique solution $y_1 \in E_1$, and from Theorem (4) the solution of (14) is **UHR** stable.
For $t \in \mathcal{J}_2$, the problem (12) can be written in the following way

$$\begin{cases} {}^C\mathcal{D}_{1+}^{\frac{9}{5}}y(t) = \dfrac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{\frac{9}{5}}y(t)|)}, \\[8pt] y(1) = 0, \;\; y(2) = 0. \end{cases}$$

$$(15)$$

We see that

$$\dfrac{(b_2^{1-\delta}-b_1^{1-\delta})(b_2-b_1)^{u_2-1}}{(1-\delta)\Gamma(u_2)} \left(2K + \dfrac{2L(b_2-b_1)^{u_2}}{\Gamma(u_2+1)} \right)$$

$$= \dfrac{2^{\frac{2}{3}}-1}{\frac{2}{3}\Gamma(\frac{9}{5})} \dfrac{2}{e+5} \left(1 + \dfrac{1}{\Gamma(\frac{14}{5})} \right) \simeq 0.3913 < 1$$

Thus, the condition (9) is satisfied.
Also

$$\begin{aligned} I_{1+}^{u_2}\psi(t) &= \dfrac{1}{\Gamma(\frac{9}{5})} \int_1^t (t-s)^{\frac{4}{5}}s^{\frac{1}{2}}ds \\[8pt] &\leq \dfrac{1}{\Gamma(\frac{9}{5})} \int_1^t (t-s)^{\frac{4}{5}}ds \\[8pt] &\leq \dfrac{5}{9\Gamma(\frac{9}{5})}\psi(t) \\[8pt] &:= \lambda_\psi \psi(t). \end{aligned}$$

Therefore, the condition (H3) is satisfied with $\psi(t) = t^{\frac{1}{2}}$ and $\lambda_\psi = \frac{5}{9\Gamma(\frac{9}{5})}$.
Taking into account of Theorem (3), the problem (15) has a unique solution $\widetilde{y}_2 \in E_2$, and from Theorem (4) the equation (15) has **UHR** stability.
It is known that

$$y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1 \\ \widetilde{y}_2(t), & t \in \mathcal{J}_2. \end{cases}$$

Hence, by considering definition (3), the boundary value problem (12) has got a unique solution

$$y(t) = \begin{cases} y_1(t), & t \in \mathcal{J}_1, \\ y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1, \\ \widetilde{y}_2(t), & t \in \mathcal{J}_2. \end{cases} \end{cases}$$

Eventually, according to Theorem (4), the equation (12) is **UHR** stable with respect to ψ.

6. Conclusion

We study some qualitative properties for a class of nonlinear fractional boundary value problems involving variable order operators. Since the existence and uniqueness as well as stability results to variable-order equations is rarely discussed in the literature, all of the outcomes in this paper have a great deal of potential for contributing to future researches.

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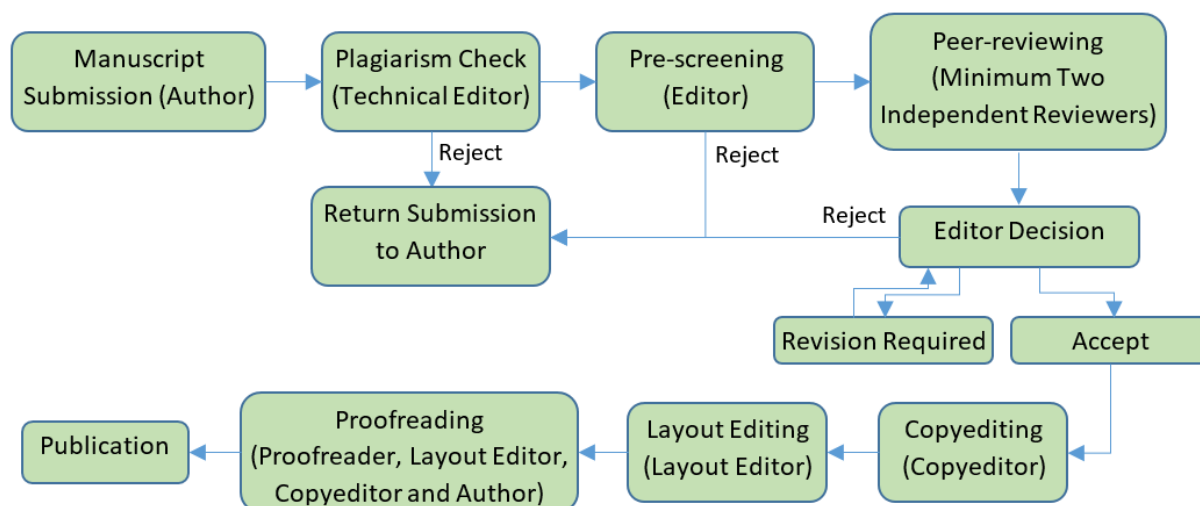

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- Researchers should strive to describe their methods clearly and unambiguously so that their findings can be confirmed by others.
- Researchers should adhere to publication requirements that submitted work is original, is not plagiarised, and has not been published elsewhere.
- Authors should take collective responsibility for submitted and published work.
- The authorship of research publications should accurately reflect individuals' contributions to the work and its reporting.
- Funding sources and relevant conflicts of interest should be disclosed.
- When an author discovers a significant error or inaccuracy in his/her own published work, it is the author's obligation to promptly notify the journal's Editor-in-Chief and cooperate with them to either retract the paper or to publish an appropriate erratum.

Reference:

Wager E & Kleinert S (2011) Responsible research publication: international standards for authors. A position statement developed at the 2nd World Conference on Research Integrity, Singapore, July 22-24, 2010. Chapter 50 in: Mayer T & Steneck N (eds) Promoting Research Integrity in a Global Environment.

Imperial College Press / World Scientific Publishing, Singapore (pp 309-16). (ISBN 978-981-4340-97-7) [Link].

Basic principles to which peer reviewers should adhere

Peer review in all its forms plays an important role in ensuring the integrity of the scholarly record. The process depends to a large extent on trust and requires that everyone involved behaves responsibly and ethically. Peer reviewers play a central and critical part in the peer-review process as the peer review assists the Editors in making editorial decisions and, through the editorial communication with the author, may also assist the author in improving the manuscript.

Peer reviewers should:

- respect the confidentiality of peer review and not reveal any details of a manuscript or its review, during or after the peer-review process, beyond those that are released by the journal;
- not use information obtained during the peer-review process for their own or any other person's or organization's advantage, or to disadvantage or discredit others;
- only agree to review manuscripts for which they have the subject expertise required to carry out a proper assessment and which they can assess within a reasonable time-frame;
- declare all potential conflicting interests, seeking advice from the journal if they are unsure whether something constitutes a relevant conflict;
- not allow their reviews to be influenced by the origins of a manuscript, by the nationality, religion, political beliefs, gender or other characteristics of the authors, or by commercial considerations;
- be objective and constructive in their reviews, refraining from being hostile or inflammatory and from making libellous or derogatory personal comments;
- acknowledge that peer review is largely a reciprocal endeavour and undertake to carry out their fair share of reviewing, in a timely manner;
- provide personal and professional information that is accurate and a true representation of their expertise when creating or updating journal accounts.

Reference:

Homes I (2013). COPE Ethical Guidelines for Peer Reviewers, March 2013, v1 [Link].