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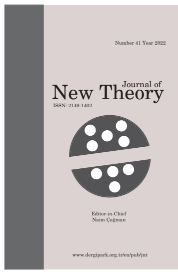

Existence, Uniqueness, and Stability of Solutions to Variable Fractional Order Boundary Value Problems

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Research Article

Abstract — This paper investigates the sufficient conditions for the existence and uniqueness of a class of Riemann-Liouville fractional differential equations of variable order with fractional boundary conditions. The problem is converted into differential equations of constant orders by combining the concepts of generalized intervals and piecewise constant functions. We derive the required conditions for ensuring the uniqueness of the problem in order to utilize the Banach fixed point theorem. The stability of the obtained solution in the Ulam-Hyers-Rassias (UHR) sense is also investigated, and we finally provide an illustrative example.

Keywords — *Existence, uniqueness, stability, boundary value problems, fractional calculus*

Mathematics Subject Classification (2020) — 26A33, 34D20

1. Introduction

Fractional calculus, which includes differentiation and integration has a history dating back over three centuries [1]. It has extended integration and differentiation operations to any fractional orders that might take any real or complex value. Thus, it is possible to think of the order of the fractional integrals and derivatives as a function of time or another variable. In this context, Samko and Ross examined the first study regarding the idea of variable order (VO) differentiation in [2, 3]. Based on using the R-L derivative and the Fourier transform, they have defined and interpreted the integration and differentiation of functions to a variable order $(\frac{d}{dx})^{\alpha(t)} f(x)$. The notion of variable and distributed order fractional operators is then developed by Lorenzo and Hartley. They reviewed the VO fractional operator research results and then studied the concepts of variable order fractional operators in various forms [4, 5].

The memory and heredity aspects of numerous physical processes and events can be used to characterize by the variable order fractional operators thanks to their non-stationary power-law kernel. As a result, fractional calculus with variable order was used as a prospective option to provide an appropriate mathematical framework for precisely modelling complicated physical systems and processes. Having followed that, VO-FDEs have attracted increasing attention, owing to their compatibility with describing a wide range of phenomena, including anomalous diffusion, medicine, viscoelasticity, control system, and many other branches of physics and engineering, to name a few [6–13]. Many publications have been devoted to finding numerical solutions for fractional differential equations of VO due to the difficulty in obtaining explicit solutions. See also [14–18] and the references therein. Nonetheless,

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some recent publications discuss the existence, uniqueness, and stability features of variable fractional order differential equations [19–32].

We aim to study following fractional boundary value problem (BVP) of variable order in Riemann-Liouville sense (VORLFDE) with fractional variable order boundary conditions as well.

$$\begin{cases} \mathcal{D}^{w(t)}y(t) = g(t, y(t)), & t \in \mathcal{J} \\ I^{2-w(t)}y(0) = \alpha_0 I^{2-w(t)}y(a), & \mathcal{D}^{w(t)-1}y(0) = \alpha_1 \mathcal{D}^{w(t)-1}y(a) \end{cases} \tag{1}$$

where $\mathcal{J} = [0, a]$, $0 < a < \infty$, $w(t) : \mathcal{J} \to (1, 2]$ is the variable order of the fractional derivatives, $g : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a given function and $\mathcal{D}^{w(t)}$, $I^{w(t)}$ denotes the Riemann-Liouville fractional derivative and integral of order $w(t)$ respectively and α_0, α_1 real numbers such that $\alpha_0 \neq 1$ and $\alpha_1 \neq 1$.

We will be concerned with the existence and uniqueness of solution of problem 1 and further study the stability of the obtained solution of problem 1 in the Ulam-Hyers-Rassias (UHR) sense.

2. Mathematical Preliminaries

This section introduces several important notions and lemmas that are required to grasp the main theorems covered in the next parts. We also present additional features for variable order operators.

Let $C(\mathcal{J}, R)$ be the set of all continuous real-valued functions from $\mathcal{J}$ into $\mathbb{R}$. Setting the standard norm $\|\varkappa\| = sup\{|\varkappa(t)| : t \in \mathcal{J}\}$ for an element in $C(\mathcal{J}, \mathbb{R})$, then $C(\mathcal{J}, \mathbb{R})$ has become a Banach space with such a norm.

For $-\infty < t_1 < t_2 < +\infty$, we consider the mappings $w(t) : [t_1, t_2] \to (0, +\infty)$ and $\theta(t) : [t_1, t_2] \to (n-1, n)$. Then, the left Riemann-Liouville fractional integral (RLFI) of variable-order $w(t)$ for function $y(t)$ is given [3] by

$$I_{t_1^+}^{w(t)}y(t) = \int_{t_1}^{t} \frac{(t-s)^{w(t)-1}}{\Gamma(w(t))} y(s)ds, \;\; t > a_1 \tag{2}$$

and the left Riemann-Liouville fractional derivative (RLFD) of variable-order $\theta(t)$ for function $y(t)$ is defined by

$$\mathcal{D}_{t_1^+}^{\theta(t)}y(t) = \left(\frac{d}{dt}\right)^n I_{t_1^+}^{n-\theta(t)}y(t) = \left(\frac{d}{dt}\right)^n \int_{t_1}^{t} \frac{(t-s)^{n-\theta(t)-1}}{\Gamma(n-\theta(t))} y(s)ds, \quad t > t_1 \tag{3}$$

As expected, RLFI and RLFD coincide with the conventional Riemann-Liouville fractional derivative and integral, respectively [1,3], when replacing constant values by $w(t)$ and $\theta(t)$.

Remark 2.1. [28] It should be emphasized that for R-L fractional integrals with constant orders, the semi-group property is satisfied, but not for those with variable orders, i.e.,

$$I_{t_1^+}^{w(t)} I_{t_1^+}^{\theta(t)} y(t) \neq I_{t_1^+}^{w(t)+\theta(t)} y(t)$$

Definition 2.2. [18] Let I be a subset of $\mathbb{R}$. Then we define the followings:

- If the set I is an interval, a point or an empty set, it is referred to as a generalized interval.
- If each x in I lies in precisely one of the generalized intervals E in $\mathcal{P}$, then the finite set $\mathcal{P}$ of generalized intervals is known as a partition of I
- If for any $E \in \mathcal{P}$, g is constant on E, the function $g : I \to \mathbb{R}$ is said to be piecewise constant with regard to partition $\mathcal{P}$ of I

Theorem 2.3. [34] Suppose E is a Banach space. If $T : E \to E$ is a completely continuous operator and $\Omega = \{x \in E : x = \eta Tx, 0 < \eta < 1\}$ is bounded, then T has a fixed point in E.

Definition 2.4. [33] BVP 1 is said to be Hyers-Ulam-Rassias stable (UHR) with regard to the function $\kappa \in C(\mathcal{J}, \mathbb{R}_+)$ if a constant $c_g > 0$ exists such that for any $\epsilon > 0$ and for each function $z \in C(\mathcal{J}, \mathbb{R})$ satisfying

$$|\mathcal{D}_{0^+}^{w(t)} z(t) - g(t, z(t))| \leq \epsilon\kappa(t), \ t \in \mathcal{J} \tag{4}$$

there exists a solution $y \in C(\mathcal{J}, \mathbb{R})$ of BVP 1 with

$$|z(t) - y(t)| \leq c_g \epsilon\kappa(t), \ t \in \mathcal{J}$$

3. Existence of Solutions

Let us proceed by stating the following hypothesis:

(H1) Assume that $\{a_k\}_{k=0}^n$ is the finite sequence of points such that $0 = a_0 < a_k < a_n = a, \ k = 1, \ldots, n-1$ where $n \in \mathbb{N}$. Let $\mathcal{J}_k := (a_{k-1}, a_k], \ k = 1, 2, \ldots, n$. Then, $\mathcal{P} = \cup_{k=1}^n \mathcal{J}_k$ is a partition of the interval $\mathcal{J}$.

For each $m = 1, \ldots, n$, the symbol $E_m = C(\mathcal{J}_m, \mathbb{R})$, indicates the Banach space of continuous functions $y : \mathcal{J}_m \to \mathbb{R}$ equipped with sup-norm $\|y\|_{E_m} = \sup_{t \in \mathcal{J}_m} |y(t)|$.

Let $w(t) : \mathcal{J} \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

$$w(t) = \sum_{m=1}^{n} w_m I_m(t)$$

where $1 < w_m \leq 2$ are constants, and I_m stands for the indicator of the interval $\mathcal{J}_m, \ m = 1, 2, \ldots, n$, that is,

$$I_m(t) = \begin{cases} 1, & t \in \mathcal{J}_m \\ 0, & elsewhere \end{cases}$$

For any $t \in \mathcal{J}_m, \ m = 1, \ldots, n$, one can represent R-L fractional variable-order derivative $w(t)$ of the function $y(t) \in C(\mathcal{J}, \mathbb{R})$, defined by (3), as the sum of left R-L fractional derivatives of integer orders $w_k, \ k = 1, \ldots, m$

$$\begin{aligned} \mathcal{D}_{0^+}^{w(t)} y(t) &= \frac{1}{\Gamma(2 - w(t))} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-w(t)} y(s) ds \\ &= \frac{1}{\Gamma(2 - w(t))} \left(\sum_{k=1}^{m-1} \frac{d^2}{dt^2} \int_{a_{k-1}}^{a_k} (t-s)^{1-w_k} y(s) ds + \frac{d^2}{dt^2} \int_{a_{m-1}}^{t} (t-s)^{1-w_m} y(s) ds \right) \end{aligned} \tag{5}$$

As a consequence, BVP 1 can be expressed on $\mathcal{J}_m$ for each $m = 1, \ldots, n$ in the manner shown below

$$\frac{1}{\Gamma(2 - w(t))} \Big(\sum_{k=1}^{m-1} \frac{d^2}{dt^2} \int_{a_{k-1}}^{a_k} (t-s)^{1-w_k} y(s) ds + \frac{d^2}{dt^2} \int_{a_{m-1}}^{t} (t-s)^{1-w_m} y(s) ds \Big) = g(t, y(t)) \tag{6}$$

Let the function $\tilde{y} \in C(\mathcal{J}_m, \mathbb{R})$ be such that $\tilde{y}(t) \equiv 0$ on $t \in [0, a_{m-1}]$ and it solves integral Equation 6. Then, it is reduced to

$$\mathcal{D}_{a_{m-1}^+}^{w_m} \tilde{y}(t) = g(t, \tilde{y}(t)), \quad t \in \mathcal{J}_m$$

We consider the auxiliary BVP given below for integer order Riemann-Liouville fractional differential equations while regarding the aforementioned statement above for any $m = 1, 2, \ldots, n$.

$$\begin{cases} \mathcal{D}_{a_{m-1}^+}^{w_m} y(t) = g(t, y(t)), & t \in \mathcal{J}_m \\ \mathcal{D}_{a_{m-1}^+}^{w_m - 2} y(a_{m-1}) = \alpha_0 \mathcal{D}_{a_{m-1}^+}^{w_m - 2} y(a_m), & \mathcal{D}_{a_{m-1}^+}^{w_m - 1} y(a_{m-1}) = \alpha_1 \mathcal{D}_{a_{m-1}^+}^{w_m - 1} y(a_m) \end{cases} \tag{7}$$

Lemma 3.1. Let $\alpha_0, \alpha_1 \neq 1$, $g \in C(\mathcal{J}_m \times \mathbb{R}, \mathbb{R})$ for $m = 1, \ldots, n$, and $\gamma \in (0,1)$ be a number such that $t^\gamma g \in C(\mathcal{J}_m \times \mathbb{R}, \mathbb{R})$.

Then, the function $x \in E_m$ satisfies problem 7 iff y solves the integral equation

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(w_m)} \int_{a_{m-1}}^{t} (t-s)^{w_m-1} g(s,x(s))ds + \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)} \int_{a_{m-1}}^{a_m} g(s,x(s))ds \\ & \frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)} \int_{a_{m-1}}^{a_m} [(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s] g(s,x(s))ds \end{aligned} \quad (8)$$

Proof. Let $x \in E_m$ be a solution of BVP 7. Using the operator $I^{w_m}_{a^+_{m-1}}$ on each sides of BVP 7, we find

$$x(t) = \lambda_1 t^{w_m-1} + \lambda_0 t^{w_m-2} + I^{w_m}_{a^+_{m-1}} g(t,x(t)) \quad (9)$$

where λ_0, λ_1 are constants.

Using funciton 9, we have

$$\mathcal{D}^{w_m-1} x(t) = \lambda_1 \Gamma(w_m) + I^1 g(t,x(t))$$

In view of assumptions on the function g and by the boundary condition

$$\mathcal{D}^{w_m-1}_{a^+_{m-1}} x(a_{m-1}) = \alpha_1 \mathcal{D}^{w_m-1}_{a^+_{m-1}} x(a_m)$$

we conclude that

$$\lambda_1 = \frac{\alpha_1}{(1-\alpha_1)\Gamma(w_m)} \int_{a_{m-1}}^{a_m} g(s,x(s))ds$$

since $I^{2-w_m}(t^{w_m-1}) = \Gamma(w_m)t$ and $I^{2-w_m}(t^{w_m-2}) = \Gamma(w_m-1)$, from the boundary condition

$$\mathcal{D}^{w_m-2}_{a^+_{m-1}} x(a_{m-1}) = \alpha_0 \mathcal{D}^{w_m-2}_{a^+_{m-1}} x(a_m)$$

we get

$$\lambda_0 = \frac{\alpha_1(\alpha_0 a_m - a_{m-1})}{(1-\alpha_1)(1-\alpha_0)\Gamma(w_m-1)} \int_{a_{m-1}}^{a_m} g(s,x(s))ds + \frac{\alpha_0}{(1-\alpha_0)\Gamma(w_m-1)} \int_{a_{m-1}}^{a_m} (a_m - s) g(s,x(s))ds$$

Thus,

$$x(t) = \int_{a_{m-1}}^{a_m} G_m(t,s) g(s,x(s))ds$$

where $G_m(t,s)$ is Green's function defined by:

$$G_m(t,s) = \begin{cases} \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)} + \frac{t^{w_m-2}[\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)} + \frac{1}{\Gamma(w_m)}(t-s)^{w_m-1} \\ \qquad a_{m-1} \leq s \leq t \leq a_m \\ \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)} + \frac{t^{w_m-2}[\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)} \\ \qquad a_{m-1} \leq t \leq s \leq a_m \end{cases}$$

where $m = 1, 2, ..., n$.

Then, we get x that solves Integral Equation (8).

In contrast, consider $x \in E_m$ to be a solution of Integral Equation 8. We conclude that x is the solution to the BVP 7 by virtue of the continuity of function $t^\gamma g$. □

We shall demonstrate the existence result for the BVP of R-L fractional differential equations of constant order (7). The proof will be carried out by the aid of Theorem 2.3.

Theorem 3.2. Suppose that the conditions of Lemma 3.1 hold and there exists a constant $N > 0$ such that

$$t^{\gamma}|g(t,y)| \leq N, \quad \forall t \in \mathcal{J}_m, y \in \mathbb{R}$$

with $\gamma = 2 - w$. Then, BVP 7 for Riemann-Liouville fractional differential equations of integer order has at least one solution in $C_{\gamma}[a_{m-1}, a_m]$.

Proof. For any function $y \in C_{\gamma}[a_{m-1}, a_m]$, we construct the operator

$$\begin{aligned} Sy(t) &= \frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^{t}(t-s)^{w_m-1}g(s,y(s))ds + \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} g(s,y(s))ds \\ &+\frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]g(s,y(s))ds \end{aligned} \quad (10)$$

It results in immediately by the properties of fractional integrals and the continuity of function $t^{\gamma}g$ that the operator $S : C_{\gamma}[a_{m-1}, a_m] \to C_{\gamma}[a_{m-1}, a_m]$ given by equality 10 is well defined. Let

$$\begin{aligned} R_m &\geq \frac{N(a_m^{1-\gamma} - a_{m-1}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m - a_{m-1})^{w_m-1} + \Big|\frac{\alpha_1 a_m}{1-\alpha_1}\Big| + \Big|\frac{(\alpha_0 a_m - \alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big| \\ &+\Big|\frac{\alpha_0(1-\alpha_1)(w_m-1)(1-\gamma)(a_m^{2-\gamma} - a_{m-1})^{2-\gamma}}{(1-\alpha_0)(1-\alpha_1)(2-\gamma)(a_m^{1-\gamma} - a_{m-1})^{1-\gamma}}\Big|\Big] \end{aligned}$$

Consider the set

$$B_{R_m} = \{y \in C_{\gamma}[a_{m-1}, a_m], \ \|y\|_{\gamma} \leq R_m\}$$

For all $m \in \{1, 2, ..., n\}$, the ball B_{R_m} is a nonempty closed convex subset of $C_{\gamma}[a_{m-1}, a_m]$.

We are in position to examine the assumption of the Theorem 3.2 for the operator S. We shall demonstrate it in three stages.

Step 1: Let B_{R_m} be a bounded set in $C_{\gamma}[a_{m-1}, a_m]$. Hence, B_{R_m} is bounded on $C[a_{m-1}, a_m]$ and there exists a constant N such that $t^{\gamma}|g(t, y(t))| \leq N, \forall y \in B_{R_m}, t \in [a_{m-1}, a_m]$. Thus,

$$\begin{aligned} t^{\gamma}|(Sy)(t)| &\leq \frac{Nt^{\gamma}}{\Gamma(w_m)}\int_{a_m-1}^{t} s^{-\gamma}(t-s)^{w_m-1}ds + \Big|\frac{\alpha_1 N t}{(1-\alpha_1)\Gamma(w_m)}\Big|\int_{a_{m-1}}^{a_m} s^{-\gamma}ds \\ &+\Big|\frac{N}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\Big|\int_{a_{m-1}}^{a_m}\Big|\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s\Big|s^{-\gamma}ds \\ &\leq \frac{N(a_m^{1-\gamma} - a_{m-1}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m - a_{m-1})^{w_m-1} + \Big|\frac{\alpha_1 a_m}{1-\alpha_1}\Big| + \Big|\frac{(\alpha_0 a_m - \alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big| \\ &+\Big|\frac{\alpha_0(1-\alpha_1)(w_m-1)(1-\gamma)(a_m^{2-\gamma} - a_{m-1})^{2-\gamma}}{(1-\alpha_0)(1-\alpha_1)(2-\gamma)(a_m^{1-\gamma} - a_{m-1})^{1-\gamma}}\Big|\Big] \end{aligned}$$

which implies that

$$\begin{aligned} \|(Sy)\|_{\gamma} \leq & \frac{N(a_m^{1-\gamma} - a_{m-1}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m - a_{m-1})^{w_m-1} + \Big|\frac{\alpha_1 a_m}{1-\alpha_1}\Big| + \Big|\frac{(\alpha_0 a_m - \alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big| \\ &+\Big|\frac{\alpha_0(1-\alpha_1)(w_m-1)(1-\gamma)(a_m^{2-\gamma} - a_{m-1})^{2-\gamma}}{(1-\alpha_0)(1-\alpha_1)(2-\gamma)(a_m^{1-\gamma} - a_{m-1})^{1-\gamma}}\Big|\Big] \end{aligned}$$

Hence, $S(B_{R_m})$ is uniformly bounded.

Step 2: Let $t_1, t_2 \in \mathcal{J}_m, \ t_1 < t_2$and $y \in B_{R_m}$. Then, we have

$$\begin{aligned} \Big|t_1^{\gamma}(Sy)(t_1) - t_2^{\gamma}(Sy)(t_2)\Big| = & \Big|\tfrac{1}{\Gamma(w_m)}\textstyle\int_{a_{m-1}}^{t_1}\Big[t_1^{\gamma}(t_1-s)^{w_m-1} - t_2^{\gamma}(t_2-s)^{w_m-1}\Big]g(s,y(s))ds \\ &-\tfrac{1}{\Gamma(w_m)}\textstyle\int_{t_1}^{t_2} t_2^{\gamma}(t_2-s)^{w_m-1}g(s,u(s))ds + \tfrac{\alpha_1(t_1-t_2)}{(1-\alpha_1)\Gamma(w_m-1)}\textstyle\int_{a_m-1}^{a_m} g(s,y(s))ds\Big| \\ \leq & N\Big(\Big|\tfrac{1}{\Gamma(w_m)}\textstyle\int_{a_{m-1}}^{t_1}\Big[t_1^{\gamma}(t_1-s)^{w_m-1} - t_2^{\gamma}(t_2-s)^{w_m-1}\Big]ds \\ &-\tfrac{1}{\Gamma(w_m)}\textstyle\int_{t_1}^{t_2} t_2^{\gamma}(t_2-s)^{w_m-1}ds\Big| + \Big|\tfrac{\alpha_1(t_1-t_2)}{(1-\alpha_1)\Gamma(w_m-1)}\textstyle\int_{a_m-1}^{a_m} ds\Big|\Big) \end{aligned}$$

Hence, $\left|t_1^{\gamma}(Sy)(t_1)-t_2^{\gamma}(Sy)(t_2)\right| \to 0$ as $|t_1-t_2| \to 0$. Thus, $t^{\gamma}S(B_{R_m})$ is equicontinuous. Consequently, the operator S is compact.

Step3: Consider the set

$$\Omega = \{y \in \mathbb{R} \setminus y = \eta Sy, 0 < \eta < 1\}$$

and show that the set Ω is bounded. Let $y \in \Omega$, then $y = \eta Sy, 0 < \eta < 1$. For any $t \in [a_m - 1, a_m]$, we have

$$|y(t)| \leq \eta\Big[\tfrac{1}{\Gamma(w_m)}\int_{a_m-1}^{t}(t-s)^{w_m-1}|g(s,y(s))|ds + \left|\tfrac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\right|\int_{a_{m-1}}^{a_m}|g(s,y(s))|ds$$
$$+\left|\tfrac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1}\right|\int_{a_{m-1}}^{a_m}\left|\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s\right|ds\Big]$$

We have

$$\|(Sy)\|_{\gamma} \leq \frac{N(a_m^{1-\gamma}-a_{m-1}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m-a_{m-1})^{w_m-1} + \left|\frac{\alpha_1 a_m}{1-\alpha_1}\right| + \left|\frac{(\alpha_0 a_m - \alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\right|$$
$$+\left|\frac{\alpha_0(1-\alpha_1)(w_m-1)(1-\gamma)(a_m^{2-\gamma}-a_{m-1})^{2-\gamma}}{(1-\alpha_0)(1-\alpha_1)(2-\gamma)(a_m^{1-\gamma}-a_{m-1})^{1-\gamma}}\right|\Big]$$

This implies that the set Ω is bounded independently of $\eta \in (0,1)$. On account of Theorem 2.3, we find that the operator S has at least one fixed point, which follows that problem 7 possesses at least one solution. □

Consider the next assumption:

(H2) Let $g \in C(\mathcal{J} \times \mathbb{R}, \mathbb{R})$ and there exist a constant $K > 0$ such that

$$t^{\gamma}|g(t,u) - g(t,v)| \leq K|u-v|$$

for any $u,\ v \in \mathbb{R}$, $t \in \mathcal{J}$ and $\gamma = 2 - w$.

Theorem 3.3. Assume that conditions (H1) and (H2) hold. Then, problem 7 has a unique solution in $C_{\gamma}[a_{m-1}, a_m]$ if

$$K < \frac{1}{\rho} \tag{11}$$

where

$$\rho = \tfrac{(a_m^{1-2\gamma}-a_{m-1}^{1-2\gamma})}{(1-2\gamma)\Gamma(w_m)}\Big[a_m^{\gamma}(a_m-a_{m-1})^{w_m-1} + \left|\tfrac{\alpha_1}{1-\alpha_1}\right|a_m + \left|\tfrac{(\alpha_0 a_m-\alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\right|$$
$$+\left|\tfrac{\alpha_0(1-\alpha_1)(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\right|\Big(\tfrac{(1-2\gamma)(a_m^{2-2\gamma}-a_{m-1}^{2-2\gamma})}{(2-2\gamma)(a_m^{1-2\gamma}-a_{m-1}^{1-2\gamma})}\Big)\Big]$$

PROOF. In view of (H2), for every $t \in [a_{m-1}, a_m]$, we have

$$t^{\gamma}|(Su)(t)-(Sv)(t)| \leq \tfrac{t^{\gamma}}{\Gamma(w_m)}\int_{a_m-1}^{t}(t-s)^{w_m-1}|g(s,u(s))-g(s,v(s))|ds$$
$$+\left|\tfrac{\alpha_1 t}{(1-\alpha_1)\Gamma(w_m)}\right|\int_{a_{m-1}}^{a_m}|g(s,u(s))-g(s,v(s))|ds$$
$$+\left|\tfrac{1}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\right|\int_{a_{m-1}}^{a_m}\left|\alpha_0 a_m-\alpha_1 a_{m-1}-\alpha_0(1-\alpha_1)s\right||g(s,u(s))-g(s,v(s))|ds$$
$$\leq K\Big[\tfrac{t^{\gamma}}{\Gamma(w_m)}\int_{a_m-1}^{t}s^{-\gamma}(t-s)^{w_m-1}|u(s)-v(s)|ds + \left|\tfrac{\alpha_1 t}{(1-\alpha_1)\Gamma(w_m)}\right|\int_{a_{m-1}}^{a_m}s^{-\gamma}|u(s)-v(s)|ds$$
$$+\left|\tfrac{1}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\right|\int_{a_{m-1}}^{a_m}s^{-\gamma}\left|\alpha_0 a_m-\alpha_1 a_{m-1}-\alpha_0(1-\alpha_1)s\right||u(s)-v(s)|ds\Big]$$

By the definition of $\|.\|_\gamma$, we obtain

$$\|(Su)(t)-(Sv)(t)\|_\gamma \le K\Big[\tfrac{t^\gamma}{\Gamma(w_m)}\int_{a_m-1}^{t} s^{-2\gamma}(t-s)^{w_m-1}ds + \Big|\tfrac{\alpha_1 t}{(1-\alpha_1)\Gamma(w_m)}\Big|\int_{a_{m-1}}^{a_m} s^{-2\gamma}ds$$
$$+\Big|\tfrac{1}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\Big|\int_{a_{m-1}}^{a_m} s^{-2\gamma}\Big|\alpha_0 a_m-\alpha_1 a_{m-1}-\alpha_0(1-\alpha_1)s\Big|ds\Big]\|u-v\|_\gamma$$
$$\le \tfrac{K(a_m^{1-2\gamma}-a_{m-1}^{1-2\gamma})}{(1-2\gamma)\Gamma(w_m)}\Big[a_m^\gamma(a_m-a_{m-1})^{w_m-1}+\Big|\tfrac{\alpha_1}{1-\alpha_1}\Big|a_m+\Big|\tfrac{(\alpha_0 a_m-\alpha_1 a_{m-1})(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big|$$
$$+\Big|\tfrac{\alpha_0(1-\alpha_1)(w_m-1)}{(1-\alpha_0)(1-\alpha_1)}\Big|\Big(\tfrac{(1-2\gamma)(a_m^{2-2\gamma}-a_{m-1}^{2-2\gamma})}{(2-2\gamma)(a_m^{1-2\gamma}-a_{m-1}^{1-2\gamma})}\Big)\Big]\|u-v\|_\gamma$$

□

From the above estimate, it follows by condition 11 that the operator S is a contraction. As a result of the Banach fixed point theorem, we may derive that S has a unique fixed point, which corresponds to a unique solution to problem 7.

Now, we will prove the existence result for problem 1.

Theorem 3.4. Let the assumptions (H1), (H2) and inequality 11 hold for all $m \in \{1,2,\dots,n\}$. Then, problem 1 has unique solution in $C_\gamma[0,a]$.

PROOF. For each $m \in \{1,2,\dots,n\}$, owing to Theorem 3.3 the BVP 7 for R-L fractional differential equations of integer order possesses unique solution $\widetilde{y}_m \in C_\gamma[a_{m-1},a_m]$. For any $m \in \{1,2,\dots,n\}$, we define the function

$$y_m = \begin{cases} 0, & t\in[0,a_{m-1}] \\ \widetilde{y}_m, & t\in\mathcal{J}_m \end{cases} \tag{12}$$

As a result, the function $y_m \in C_\gamma[a_{m-1},a_m]$ satisfies the integral problem 6 on $\mathcal{J}_m$, implying that $y_m(0)=0,\ y_m(a_m)=\widetilde{y}_m(a_m)=0$ and solves problem 6 for $t\in\mathcal{J}_m,\quad m\in\{1,2,\dots,n\}$. Then, the function

$$y(t) = \begin{cases} y_1(t), t\in\mathcal{J}_1 \\ y_2(t) = \begin{cases} 0, & t\in\mathcal{J}_1 \\ \widetilde{y}_2, & t\in\mathcal{J}_2 \end{cases} \\ \vdots \\ y_n(t) = \begin{cases} 0, & t\in[0,a_{n-1}] \\ \widetilde{y}_n, & t\in\mathcal{J}_n \end{cases} \end{cases}$$

is a solution of BVP 1 in $C_\gamma[0,a]$. □

4. Ulam-Hyers-Rassias Stability of VORLFDE

Theorem 4.1. Let the conditions (H1), (H2) and inequality 11 be satisfied. Further assume that

(H3) $\kappa \in C(\mathcal{J},\mathbb{R}_+)$ is increasing and there exists $\lambda_\kappa > 0$ such that

$$I_{a_{m-1}^+}^{w_m}\kappa(t) \le \lambda_\kappa\ \kappa(t)$$

hold for $t\in\mathcal{J}_m$, $m=1,2,\dots$. Then, BVP 1 is UHR stable with respect to κ.

PROOF. Let $\epsilon>0$ be an arbitrary number and the function $z(t)$ from $C(\mathcal{J},\mathbb{R})$ satisfy inequality 4. For any $m\in\{1,2,\dots,n\}$ we define the functions $z_1(t)\equiv z(t),\ t\in[0,a_1]$ and for $m=2,3,\dots,n$:

$$z_m(t) = \begin{cases} 0, & t\in[0,a_{m-1}] \\ z(t), & t\in\mathcal{J}_m \end{cases}$$

For any $m\in\{1,2,\dots,n\}$ according to equality 3 for $t\in\mathcal{J}_m$ we get

$$\mathcal{D}_{0^+}^{w(t)} z_m(t) = \frac{1}{\Gamma(2-w(t))}\frac{d^2}{dt^2}\int_{a_{m-1}}^{t}(t-s)^{1-w_m}\frac{z(s)}{s}ds \tag{13}$$

we take the RLFI $I^{w_m}_{a_{m-1}^+}$ of both sides of the inequality 4, apply (H3) and obtain

$$\begin{aligned}\Big| z_m(t) - \tfrac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t (t-s)^{w_m-1} g(s,z_m(s))ds - \tfrac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} g(s,z_m(s))ds \\ - \tfrac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]g(s,z_m(s))ds\Big| &\le \epsilon\, I^{w_m}_{a_{m-1}^+}\kappa(t) \\ &\le \epsilon\, \lambda_\kappa\, \kappa(t)\end{aligned}$$

According to Theorem 3.4, BVP 1 has a solution $y \in C(\mathcal{J},\mathbb{R})$ defined by $y(t) = y_m(t)$ for $t \in \mathcal{J}_m,\ m = 1,2,\ldots,n$, where

$$y_m = \begin{cases} 0, & t \in [0, a_{m-1}] \\ \widetilde{y}_m, & t \in \mathcal{J}_m \end{cases} \tag{14}$$

and $\widetilde{y}_m \in E_m$ is a solution of problem 7. According to Lemma 3.1, the integral equation

$$\begin{aligned}\widetilde{y}_m(t) = &\frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t (t-s)^{w_m-1} g(s,\widetilde{y}_m(s))ds + \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} g(s,\widetilde{y}_m(s))ds \\ &+ \frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]g(s,\widetilde{y}_m(s))ds\end{aligned} \tag{15}$$

holds. Let $t \in \mathcal{J}_m$ where $m \in \{1,2,\ldots,n\}$. Then, we obtain

$$\begin{aligned}|z(t) - y(t)| &= |z(t) - y_m(t)| \\ &= |z_m(t) - \widetilde{y}_m(t)| \\ &\le \Big| z_m(t) - \frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t (t-s)^{w_m-1} g(s,z_m(s))ds - \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} g(s,z_m(s))ds \\ &\quad - \frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]g(s,z_m(s))ds\Big| \\ &\quad + \frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t (t-s)^{w_m-1}\Big|g(s,z_m(s)) - g(s,\widetilde{y}_m(s))\Big|ds \\ &\quad + \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m}\Big|g(s,z_m(s)) - g(s,\widetilde{y}_m(s))\Big|ds \\ &\quad + \frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]\Big|g(s,z_m(s)) - g(s,\widetilde{y}_m(s))\Big|ds \\ &\le \lambda_\kappa\,\epsilon\,\kappa(t) + \frac{1}{\Gamma(w_m)}\int_{a_{m-1}}^t s^{-\gamma}(t-s)^{w_m-1}(K|z_m(s) - \widetilde{y}_m(s)|)ds \\ &\quad + \frac{\alpha_1 t^{w_m-1}}{(1-\alpha_1)\Gamma(w_m)}\int_{a_{m-1}}^{a_m} s^{-\gamma}(K|z_m(s) - \widetilde{y}_m(s)|)ds \\ &\quad + \frac{t^{w_m-2}}{(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}\int_{a_{m-1}}^{a_m}[(\alpha_0 a_m - \alpha_1 a_{m-1} - \alpha_0(1-\alpha_1)s]s^{-\gamma}(K|z_m(s) - \widetilde{y}_m(s)|)ds \\ &\le \lambda_\kappa\,\epsilon\,\kappa(t) + \frac{(t-a_{m-1})^{w_m-1}(t^{1-\gamma} - {a_{m-1}}^{1-\gamma})}{(1-\gamma)\Gamma(w_m)}(K\|z_m - \widetilde{y}_m\|_{E_m}) \\ &\quad + \frac{\alpha_1 t^{w_m-1}(a_m^{1-\gamma} - {a_{m-1}}^{1-\gamma})}{(1-\gamma)(1-\alpha_1)\Gamma(w_m)}(K\|z_m - \widetilde{y}_m\|_{E_m}) \\ &\quad + \frac{t^{w_m-2}a_{m-1}^{-\gamma}(a_m - a_{m-1})}{2(1-\alpha_0)(1-\alpha_1)\Gamma(w_m-1)}[2(\alpha_0 a_m - \alpha_1 a_{m-1}) - \alpha_0(1-\alpha_1)(a_m - a_{m-1})](K\|z_m - \widetilde{y}_m\|_{E_m}) \\ &\le \lambda_\kappa\,\epsilon\,\kappa(t) + \eta\|z - y\|_{\mathcal{J}}\end{aligned}$$

Then,

$$\|z-y\|_{\mathcal{J}}\Big(1-\eta\Big) \quad \leq \quad \lambda_{\kappa}\ \epsilon\ \kappa(t)$$

or for any $t \in \mathcal{J}$

$$|z(t)-y(t) \leq \|z-y\|_{\mathcal{J}} \leq \frac{\lambda_{\kappa}}{1-\eta}\epsilon\kappa(t)$$

Therefore, BVP 1 is UHR stable with respect to κ. □

5. Example

Let $\mathcal{J} := [0,4]$, $a_0 = 0$, $a_1 = 2$, $a_2 = 4$. Consider the following fractional problem with fractional boundary conditions of variable order

$$\begin{cases} \mathcal{D}_{0^+}^{w(t)}y(t) = \frac{\sin y(t)+2\cos(t)}{t^2}, \quad t \in \mathcal{J}, \\ \mathcal{D}_{0^+}^{w(t)-2}y(0) = \alpha_0\mathcal{D}_{0^+}^{w(t)-2}y(4), \ \ \mathcal{D}_{0^+}^{w(t)-1}y(0) = \alpha_1\mathcal{D}_{0^+}^{w(t)-2}y(4) \end{cases} \tag{16}$$

where

$$w(t) = \begin{cases} \frac{3}{2}, & t \in \mathcal{J}_1 := [0,2] \\ \frac{10}{9}, & t \in \mathcal{J}_2 :=]2,4] \end{cases} \tag{17}$$

Denote

$$g(t,y) = \frac{\sin y + 2\cos(t)}{t^2}, \ (t,y) \in [0,4] \times \mathbb{R}$$

Taking into account function 17, we can construct two auxiliary BVPs by means of problem 7 for Riemann-Liouville fractional differential equations of integer order

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{3}{2}}y(t) = \frac{\sin y(t)+2\cos(t)}{t^2}, \quad t \in \mathcal{J}_1, \\ \mathcal{D}_{0^+}^{\frac{-1}{2}}y(0) = \alpha_0\mathcal{D}_{0^+}^{\frac{-1}{2}}y(2), \quad \mathcal{D}_{0^+}^{\frac{1}{2}}y(0) = \alpha_1\mathcal{D}_{0^+}^{\frac{1}{2}}y(2) \end{cases} \tag{18}$$

and

$$\begin{cases} \mathcal{D}_{2^+}^{\frac{10}{9}}y(t) = \frac{\sin y(t)+2\cos(t)}{t^2}, \quad t \in \mathcal{J}_2, \\ \mathcal{D}_{2^+}^{\frac{-8}{9}}y(2) = \alpha_0\mathcal{D}_{2^+}^{\frac{-8}{9}}y(4), \quad \mathcal{D}_{2^+}^{\frac{1}{9}}y(2) = \alpha_1\mathcal{D}_{2^+}^{\frac{1}{9}}y(4) \end{cases} \tag{19}$$

For $m = 1$. Clearly,

$$t^{\gamma}\Big|g(t,y)\Big| \quad \leq \quad t^{\frac{1}{2}}\Big|\frac{\sin y(t)+2\cos(t)}{t^2}\Big| \leq \frac{3}{2\sqrt{2}} = N$$

Let $\kappa(t) = t^{\frac{1}{2}}$. Then, we obtain

$$\begin{aligned} I_{0^+}^{w_1}\kappa(t) \quad &= \quad \tfrac{1}{\Gamma(\frac{3}{2})}\textstyle\int_0^t (t-s)^{\frac{1}{2}}s^{\frac{1}{2}}ds \\ &\leq \quad \tfrac{\sqrt{2}}{\Gamma(\frac{3}{2})}\textstyle\int_0^t (t-s)^{\frac{1}{2}}ds \\ &\leq \quad \tfrac{2\sqrt{2}}{3\Gamma(\frac{3}{2})}\kappa(t) := \lambda_{\kappa(t)}\kappa(t) \end{aligned}$$

where $\lambda_{\kappa} = \frac{2\sqrt{2}}{3\Gamma(\frac{3}{2})}$. Thus, it implies that condition (H3) holds.

By Theorem 3.2, BVP 18 has a solution $\widetilde{y}_1 \in E_1$. For $m = 2$. Clearly,

$$t^{\gamma}\Big|g(t,y)\Big| \quad \leq \quad t^{\frac{8}{9}}\Big|\frac{\sin y(t)+2\cos(t)}{t^2}\Big| \leq 3.4^{\frac{-10}{9}} = N$$

Let $\kappa(t) = t^{\frac{1}{2}}$. Then, we obtain

$$\begin{aligned} I_{2^+}^{w_2}\kappa(t) &= \frac{1}{\Gamma(\frac{10}{9})}\int_2^t (t-s)^{\frac{1}{9}} s^{\frac{1}{2}} ds \\ &\leq \frac{2}{\Gamma(\frac{10}{9})}\int_2^t (t-s)^{\frac{1}{9}} ds \\ &\leq \frac{9}{5\Gamma(\frac{10}{9})}\kappa(t) := \lambda_{\kappa(t)}\kappa(t) \end{aligned}$$

where $\lambda_\kappa = \frac{9}{5\Gamma(\frac{10}{9})}$. Thus, condition (H3) is satisfied.

According to Theorem 3.2, BVP 19 possesses a solution $\widetilde{y}_2 \in E_2$. Thus, according to Theorem 3.4 the BVP 16 has a solution

$$y(t) = \begin{cases} \widetilde{y}_1(t), & t \in \mathcal{J}_1 \\ y_2(t), & t \in \mathcal{J}_2 \end{cases}$$

where

$$y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1 \\ \widetilde{y}_2(t), & t \in \mathcal{J}_2 \end{cases}$$

In view of Theorem 4.1, BVP 16 is also UHR stable with respect to κ.

6. Conclusion

This study examines the necessary and sufficient conditions for the existence and uniqueness of a class of variable order differential equations with fractional boundary conditions. Combining the ideas of generalized intervals and piecewise constant functions, the problem is transformed into differential equations of constant orders. Using the Banach fixed point theorem, we develop the conditions for guaranteeing the problem's uniqueness. We also discuss the stability of the obtained solution in the Ulam-Hyers-Rassias (UHR) sense.

For the future investigations, one can study to get similar results by considering systems of differential equations as well as employing variable order fractional operators in Caputo's sense. Moreover, the obtained results may be extended to variable order fractional boundary or initial value problems in which the non-linear term may have discontinuities at some interior points.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All authors declare no conflict of interest.

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