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Some generalizations of comparison results for fractional differential equations

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ABSTRACT

In this paper, by using the technique of upper and lower solutions together with the theory of strict and nonstrict fractional differential inequalities involving Riemann–Liouville differential operator of order q, $0 < q < 1$, some necessary comparison results for further generalizations of several dynamical concepts are obtained. Furthermore, these results are extended to the finite systems of fractional differential equations.

1. Introduction

The subject of fractional calculus, which has a long history more than three hundred years, generalizes and includes the concepts of classical derivative and integral of integer order. Recently, the advantage and importance of employing the fractional derivative to model several physical phenomena in a variety of disciplines were realized [1–7]. Since its theory is much richer than the theory of classical ordinary differential equations, the theory of fractional differential equations have attracted considerable interest. There has been a growing interest in this new area and a significant development in the study of fractional differential equations in recent years; see the monographs of [8–10], and the survey by Agarwal et al. [11].

In the past decades, some work has been done in the field of basic theoretical concepts like differential and integral inequalities, existence and uniqueness results concerning the fractional differential equations. For some recent contributions on fractional differential equations, see [12–23] and the references therein.

We now consider the initial value problem (IVP)

$$D^q x(t) = f(t,x), \quad t \in J = [t_0, T] \quad \text{and} \quad x(t)(t-t_0)^{1-q}|_{t=t_0} = x^0, \tag{1.1}$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and D^q is Riemann–Liouville (R–L) fractional derivative of order q, $0 < q < 1$.

The corresponding Volterra fractional integral equation is defined as

$$x(t) = \frac{x^0 (t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) \mathrm{d}s. \tag{1.2}$$

Assume that $f(t,x)$ has the splitting $f(t,x) = F(t,x,x)$ where $F \in C[J \times \mathbb{R}^2, \mathbb{R}]$. Then, problem (1.1) takes the following form:

$$D^q x(t) = F(t,x,x), \qquad x(t)(t-t_0)^{1-q}|_{t=t_0} = x^0. \tag{1.3}$$

In this paper, by considering the IVP (1.3), we intend to study and develop some comparison results given in [8]. Also we shall extend this idea to the finite systems of fractional differential equations.

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2. Preliminaries

In this section, we present some basic definitions and theorems which are used throughout the paper. We first give a variety of possible definitions of lower and upper solutions relative to (1.3).

Definition 2.1. Let $v, w \in C_p[J, \mathbb{R}]$, $p = 1 - q$, $0 < q < 1$ be locally Hölder continuous with exponent $\lambda > q$ and $D^q v$ and $D^q w$ exist and $F \in C[J \times \mathbb{R}^2, \mathbb{R}]$.

Then v and w are said to be

(i) natural lower and upper solutions of (1.3) respectively if

$$D^q v \le F(t, v, v), \quad v^0 \le x^0,$$
$$D^q w \ge F(t, w, w), \quad w^0 \ge x^0, \ t \in J, \tag{2.1}$$

(ii) coupled lower and upper solutions of type I of (1.3) respectively if

$$D^q v \le F(t, v, w), \quad v^0 \le x^0,$$
$$D^q w \ge F(t, w, v), \quad w^0 \ge x^0, \ t \in J. \tag{2.2}$$

(iii) coupled lower and upper solutions of type II of (1.3) respectively if

$$D^q v \le F(t, w, v), \quad v^0 \le x^0,$$
$$D^q w \ge F(t, v, w), \quad w^0 \ge x^0, \ t \in J. \tag{2.3}$$

(iv) coupled lower and upper solutions of type III of (1.3) respectively if

$$D^q v \le F(t, w, w), \quad v^0 \le x^0,$$
$$D^q w \ge F(t, v, v), \quad w^0 \ge x^0, \ t \in J. \tag{2.4}$$

where $v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t - t_0)^{1-q}|_{t=t_0}$.

Lemma 2.1. *Let $m \in C_p([t_0, T], R)$ be locally Hölder continuous with exponent $\lambda > q$ and for any $t_1 \in (t_0, T]$, we have*

$$m(t_1) = 0 \quad \text{and} \quad m(t) \le 0 \quad \text{for } t_0 \le t \le t_1. \tag{2.5}$$

Then it follows that

$$D^q m(t_1) \ge 0. \tag{2.6}$$

Proof. For the proof please see [8]. □

The explicit solution of the nonhomogeneous linear fractional differential equation involving R–L fractional differential operator of order q $(0 < q < 1)$ is necessary for further development of our main results. So, we now consider the nonhomogeneous IVP for linear fractional differential equation

$$D^q x = \lambda x + f(t), \qquad x^0 = x(t)(t - t_0)^{1-q}|_{t=t_0}, \tag{2.7}$$

where λ is a real number and $f \in C_p([t_0, T], \mathbb{R})$.

The equivalent Volterra fractional integral equation for $t_0 \le t \le T$, is

$$x(t) = x^0(t) + \frac{\lambda}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} x(s) \mathrm{d}s + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s) \mathrm{d}s, \tag{2.8}$$

where $x^0(t) = \frac{x^0 (t-t_0)^{q-1}}{\Gamma(q)}$.

When we apply the method of successive approximations [8] to find the solution $x(t) = x(t, t_0, x^0)$ explicitly for the given nonhomogeneous IVP (2.7), we obtain

$$x(t) = x^0 (t - t_0)^{q-1} E_{q,q}(\lambda (t - t_0)^q) + \int_{t_0}^{t} (t - s)^{q-1} E_{q,q}(\lambda (t - s)^q) f(s) \mathrm{d}s, \quad t \in [t_0, T], \tag{2.9}$$

where $E_{q,q}$ denotes the two parameter Mittag-Leffler function.

If $f(t) \equiv 0$, we get, as the solution of the corresponding homogeneous IVP

$$x(t) = x^0 (t - t_0)^{q-1} E_{q,q}(\lambda (t - t_0)^q), \quad t \in [t_0, T]. \tag{2.10}$$

Next theorem relative to strict fractional differential inequalities includes Theorem 2.3 in [12] as a special case.

Theorem 2.1. *Let $v, w \in C_p[J, \mathbb{R}]$, $p = 1-q$, $0 < q < 1$ be natural lower and upper solutions of (1.3) respectively and assume that one of the inequalities in (2.1) to be strict.*

Then

$$v^0 < w^0 \tag{2.11}$$

implies that

$$v(t) < w(t) \quad on\, J. \tag{2.12}$$

Proof. Suppose that $v(t) < w(t)$ on J is false. Then there would exist a $t_1 \in (t_0, T]$ such that

$$v(t_1) = w(t_1) \quad \text{and} \quad v(t) \le w(t), \quad t_0 \le t \le t_1.$$

Now, setting $m(t) = v(t) - w(t)$ on $[t_0, T]$, we obtain

$$m(t_1) = 0 \quad \text{and} \quad m(t) \le 0, \quad t_0 \le t \le t_1.$$

Hence, employing Lemma 2.1 we get $D^q m(t) \ge 0$. Assuming the first inequality in (2.1) being strict, we arrive at the contradiction

$$F(t_1, v(t_1), v(t_1)) > D^q v(t_1) \ge D^q w(t_1) \ge F(t_1, w(t_1), w(t_1))$$

which proves the conclusion of the theorem.

Therefore, the proof is complete. □

Remark 2.1. Instead of choosing natural lower and upper solution, one can use the coupled lower and upper solutions of all type of (1.3). Then, the conclusion of Theorem 2.1 remains valid.

3. Main results

We will give some comparison results in terms of lower and upper solutions of the problem (1.3).

Theorem 3.1. *Let v and w be natural lower and upper solutions of (1.3). Assume further that F satisfy the following condition*

$$F(t, x_1, y_1) - F(t, x_2, y_2) \le L[(x_1 - x_2) + (y_1 - y_2)], \quad L \ge 0, \tag{3.1}$$

whenever $x_1 \ge x_2$ and $y_1 \ge y_2$.

Then $v^0 \le w^0$ implies $v(t) \le w(t)$ for $t_0 \le t \le T$.

Proof. Let us first set $w_\epsilon(t) = w(t) + \epsilon\lambda(t)$ for small $\epsilon > 0$, where $\lambda(t) = (t-t_0)^{q-1}E_{q,q}(3L(t-t_0))$ and $E_{q,q}$ denotes two parameter Mittag-Leffler function. This implies that $w_\epsilon(t)(t-t_0)^{1-q}|_{t=t_0} = w_\epsilon^0 = w(t)(t-t_0)^{1-q}|_{t=t_0} + \epsilon\lambda(t)(t-t_0)^{1-q}|_{t=t_0}$.

So we find $w_\epsilon^0 = w^0 + \epsilon\lambda^0$ which gives $w_\epsilon^0 > w^0 > v^0$ and $w_\epsilon(t) > w(t)$ on J. Then, we get

$$\begin{aligned} D^q w_\epsilon(t) &= D^q w(t) + \epsilon D^q \lambda(t) \\ &\ge F(t, w, w) + 3\epsilon L\lambda(t). \end{aligned}$$

Here, we have utilized the fact that $\lambda(t)$ is the solution of IVP

$$D^q\lambda(t) = 3L\lambda(t), \qquad \lambda(t)(t-t_0)^{1-q}|_{t=t_0} = \lambda^0 > 0.$$

Also, using the inequality (2.1), we obtain

$$\begin{aligned} D^q w_\epsilon(t) &\ge F(t, w_\epsilon, w_\epsilon) - 2L(w_\epsilon - w) + 3\epsilon L\lambda(t) \\ &= F(t, w_\epsilon, w_\epsilon) + \epsilon L\lambda(t) \\ &> F(t, w_\epsilon, w_\epsilon), \quad t_0 \le t \le T. \end{aligned}$$

Therefore, employing Theorem 2.1 to the functions $v(t)$ and $w_\epsilon(t)$, we get $v(t) < w_\epsilon(t)$ on J and letting $\epsilon \to 0$, we arrive at

$$v(t) \le w(t) \quad \text{on}\, J,$$

which completes the proof. □

Remark 3.1. Theorem 3.1 includes Theorem 2.2.3 in [8] as a special case.

Theorem 3.2. *Let v and w be coupled lower and upper solutions of type* I *of* (1.3)*. Moreover, assume that F satisfy the following inequalities*

$$F(t,x_1,y)-F(t,x_2,y)\le L(x_1-x_2),\quad L\ge 0, \tag{3.2}$$

$$F(t,x,y_1)-F(t,x,y_2)\ge -L(y_1-y_2), \tag{3.3}$$

whenever $x_1 \ge x_2$ and $y_1 \ge y_2$.

Then $v^0 \le w^0$ implies $v(t) \le w(t)$ for $t_0 \le t \le T$.

Proof. To prove the conclusion of the theorem, we set, for small $\epsilon > 0$,

$$w_\epsilon(t)=w(t)+\epsilon\lambda(t)\quad\text{and}\quad v_\epsilon(t)=v(t)-\epsilon\lambda(t), \tag{3.4}$$

where $\lambda(t) = (t-t_0)^{q-1}E_{q,q}(3L(t-t_0))$ and $E_{q,q}$ is two parameter Mittag-Leffler function. It is clear that $w_\epsilon(t) > w(t)$, $v_\epsilon(t) < v(t)$ and $v_\epsilon^0 < w_\epsilon^0$.

Then using (3.2) and (3.3), we get

$$\begin{aligned}
D^q v_\epsilon(t) &= D^q v(t)-\epsilon D^q\lambda(t)\\
&\le F(t,v,w)-3\epsilon L\lambda(t)\\
&\le F(t,v,w_\epsilon)+\epsilon L\lambda(t)-3\epsilon L\lambda(t)\\
&\le F(t,v_\epsilon,w_\epsilon)+2\epsilon L\lambda(t)-3\epsilon L\lambda(t)\\
&< F(t,v_\epsilon,w_\epsilon).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
D^q w_\epsilon(t) &= D^q w(t)+\epsilon D^q\lambda(t)\\
&\ge F(t,w,v)+3\epsilon L\lambda(t)\\
&\ge F(t,w_\epsilon,v)-\epsilon L\lambda(t)+3\epsilon L\lambda(t)\\
&\ge F(t,w_\epsilon,v_\epsilon)-2\epsilon L\lambda(t)+3\epsilon L\lambda(t)\\
&> F(t,w_\epsilon,v_\epsilon).
\end{aligned}$$

Hence, employing Theorem 2.1 to $v_\epsilon(t)$ and $w_\epsilon(t)$, we get $v_\epsilon(t) < w_\epsilon(t)$ on J. Finally, taking limit as $\epsilon \to 0$, we get $v(t) \le w(t)$ on J.

Therefore, the proof is complete. □

Theorem 3.3. *Let v and w be coupled lower and upper solutions of type* II *of* (1.3)*. Moreover, F satisfy the following inequalities*

$$F(t,x,y_1)-F(t,x,y_2)\le L(y_1-y_2),\quad L\ge 0, \tag{3.5}$$

$$F(t,x_1,y)-F(t,x_2,y)\ge -L(x_1-x_2), \tag{3.6}$$

whenever $x_1 \ge x_2$ and $y_1 \ge y_2$.

Then $v^0 \le w^0$ implies $v(t) \le w(t)$ for $t_0 \le t \le T$.

Proof. For the proof, we define, as before, for small $\epsilon > 0$,

$$w_\epsilon(t)=w(t)+\epsilon\lambda(t)\quad\text{and}\quad v_\epsilon(t)=v(t)-\epsilon\lambda(t),$$

then we use the same way as in Theorem 3.2. So we omit the details. □

Theorem 3.4. *Let v and w be coupled lower and upper solutions of type* III *of* (1.3)*. Also F satisfy the following condition*

$$F(t,x_1,y_1)-F(t,x_2,y_2)\ge -L[(x_1-x_2)+(y_1-y_2)],\quad L\ge 0, \tag{3.7}$$

whenever $x_1 \ge x_2$ and $y_1 \ge y_2$.

Then $v^0 \le w^0$ implies $v(t) \le w(t)$ for $t_0 \le t \le T$.

Proof. In this case, we need to set $v_\epsilon(t) = v(t) - \epsilon\lambda(t)$. Then we proceed, as before, to get the claim of the theorem. The proof being similar is omitted.

We can generalize this idea to finite systems of fractional differential equations. For this aim, consider the following fractional differential system

$$D^q x(t)=f(t,x),\qquad x(t)(t-t_0)^{1-q}|_{t=t_0}=x^0, \tag{3.8}$$

where $f \in C[J \times \mathbb{R}^n, \mathbb{R}^n]$ and D^q is Riemann–Liouville (R–L) fractional derivative of order q, $0 < q < 1$.

Also, the corresponding systems of fractional differential equations for (1.3) is given by

$$D^q x(t)=F(t,x,x),\qquad x(t)(t-t_0)^{1-q}|_{t=t_0}=x^0, \tag{3.9}$$

where $F \in C[J \times \mathbb{R}^{2n}, \mathbb{R}^n]$.

At this point, we shall need an important property, known as quasimonotone nondecreasing relative to systems of inequalities. □

Definition 3.1. A function $f \in C[J \times \mathbb{R}^n, \mathbb{R}^n]$ is said to be possess quasimonotone nondecreasing property if for each i such that $1 \le i \le n$, $u \le v$ and $u_i = v_i$ implies $f_i(t, u) \le f_i(t, v)$.

Theorem 3.5. *Let $v, w \in C_p[J, \mathbb{R}]$, $p = 1 - q$, $0 < q < 1$ be locally Hölder continuous with exponent $\lambda > q$ and $F \in C[J \times \mathbb{R}^{2n}, \mathbb{R}^n]$. Suppose that one of the following conditions holds:*
(H_1) *v and w are natural lower and upper solutions of (3.9) respectively and $F(t, x, y)$ is quasimonotone nondecreasing in x and y for each $t \in J$ and for each i,*

$$F_i(t, x_1, y_1) - F_i(t, x_2, y_2) \le L \sum_{j=1}^{n} [(x_{1j} - x_{2j}) + (y_{1j} - y_{2j})], \quad L \ge 0, \tag{3.10}$$

whenever $x_1 \ge x_2$ and $y_1 \ge y_2$;
(H_2) *v and w are coupled lower and upper solutions of type* I *of (3.9) respectively and $F(t, x, y)$ is quasimonotone nondecreasing in x for each (t, y) and nonincreasing in y for each (t, x) and for each i,*

$$F_i(t, x_1, y) - F_i(t, x_2, y) \le L \sum_{j=1}^{n} (x_{1j} - x_{2j}), \quad L \ge 0, \text{ for each } (t, y), \tag{3.11}$$

$$F_i(t, x, y_1) - F_i(t, x, y_2) \ge -L \sum_{j=1}^{n} (y_{1j} - y_{2j}), \quad \text{for each } (t, x), \tag{3.12}$$

whenever $x_1 \ge x_2$ and $y_1 \ge y_2$;
(H_3) *v and w are coupled lower and upper solutions of type* II *of (3.9) respectively and $F(t, x, y)$ is quasimonotone nonincreasing in x for each (t, y) and nondecreasing in y for each (t, x) and for each i,*

$$F_i(t, x_1, y) - F_i(t, x_2, y) \ge -L \sum_{j=1}^{n} (x_{1j} - x_{2j}), \quad L \ge 0, \text{ for each } (t, y), \tag{3.13}$$

$$F_i(t, x, y_1) - F_i(t, x, y_2) \le L \sum_{j=1}^{n} (y_{1j} - y_{2j}), \quad \text{for each } (t, x), \tag{3.14}$$

whenever $x_1 \ge x_2$ and $y_1 \ge y_2$;
(H_4) *v and w are coupled lower and upper solutions of type* III *of (3.9) respectively and $F(t, x, y)$ is quasimonotone in x and y for each $t \in J$ and for each i,*

$$F_i(t, x_1, y_1) - F_i(t, x_2, y_2) \ge -L \sum_{j=1}^{n} [(x_{1j} - x_{2j}) + (y_{1j} - y_{2j})], \quad L \ge 0, \tag{3.15}$$

whenever $x_1 \ge x_2$ and $y_1 \ge y_2$.
Then $v^0 \le w^0$ implies $v(t) \le w(t)$ on J.

Proof. To prove the conclusion of the theorem when (H_1) holds, we define $w_\epsilon(t) = w(t) + \epsilon\lambda(t)$ for some $\epsilon > 0$ where $\epsilon = (\epsilon, \epsilon, \dots, \epsilon)$ and $\lambda(t) = (t - t_0)^{q-1} E_{q,q}[(2n + 1)(t - t_0)^q]$. So one can have $w_\epsilon(t) > w(t)$ and $w_\epsilon^0 > v^0$. Then, using (H_1), we find for each i,

$$\begin{aligned} D^q w_{\epsilon i}(t) &= D^q w_i(t) + \epsilon D^q \lambda(t) \\ &\ge F_i(t, w, w) + (2n + 1)L\epsilon\lambda(t) \\ &\ge F_i(t, w_\epsilon, w_\epsilon) - 2nL\epsilon\lambda(t) + (2n + 1)L\epsilon\lambda(t) \\ &> F_i(t, w_\epsilon, w_\epsilon). \end{aligned}$$

We shall show that $v(t) < w_\epsilon(t)$ on J, which proves the conclusion as making $\epsilon \to 0$. Suppose that $v(t) < w_\epsilon(t)$ on J is not true, then there would exist an index j, $1 \le j \le n$ and a $t_1 \in (t_0, T]$ such that

$$v_j(t_1) = w_{\epsilon j}(t_1), \quad v_j(t) \le w_{\epsilon j}(t), \ t_0 \le t \le t_1 \quad \text{and} \quad v_i(t_1) \le w_{\epsilon i}(t_1) \quad \text{for } i \ne j. \tag{3.16}$$

Setting $m(t) = v_j(t) - w_{\epsilon j}(t)$, we get $m(t_1) = 0$ and $m(t) \le 0$ for $t_0 \le t \le t_1$. We then deduce from Lemma 2.1 that $D^q m(t_1) \ge 0$ which, in view of (3.16), yields a contradiction

$$F_j(t_1, v(t_1), v(t_1)) \ge v_j(t_1) \ge w_{\epsilon j}(t_1) > F_j(t_1, w_\epsilon(t_1), w_\epsilon(t_1)) \tag{3.17}$$

proving the theorem.

The other cases can be proved by using similar discussions in this proof given above and following the proof of previous theorems with suitable changes. We omit the details.

Therefore, the proof is complete. □

4. Conclusion

In this work, some current comparison theorems in literature have been generalized in the framework of Riemann–Liouville fractional differential operators of order q, $0 < q < 1$. Here, we have utilized the technique of upper and lower solutions together with the theory of strict and nonstrict fractional differential inequalities. For the further improvement in applications of dynamical systems, we have extended these results to the finite systems of fractional differential equations.

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