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Domain of the Cesàro mean matrix in some paranormed spaces of double sequences

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Abstract. In this study, we define the paranormed sequence spaces $\widetilde{\mathcal{M}}_u(t)$, $\widetilde{\mathcal{C}}_p(t), \widetilde{\mathcal{C}}_{0p}(t)$, $\widetilde{\mathcal{C}}_r(t)$, $\widetilde{\mathcal{C}}_{bp}(t)$ and $\widetilde{\mathcal{L}}_u(t)$ of double sequences whose double Cesàro mean transforms are bounded, convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, regularly convergent and absolutely summable, respectively. Furthermore, we examine some properties of the double sequence spaces $\widetilde{\mathcal{M}}_u(t)$, $\widetilde{\mathcal{C}}_p(t), \widetilde{\mathcal{C}}_{0p}(t)$, $\widetilde{\mathcal{C}}_r(t)$, $\widetilde{\mathcal{C}}_{bp}(t)$ and $\widetilde{\mathcal{L}}_u(t)$.

Key words. Double sequence spaces, the double Cesáro matrix of order one, paranormed double sequence spaces.

Cesàro matrisinin bazı paranormlu çift dizi uzaylarındaki etki alanı

Özet. Bu çalışmada; sırasıyla double Cesàro dönüşümleri sınırlı, Pringsheim anlamında yakınsak, Pringsheim anlamında sıfıra yakınsak, hem Pringsheim anlamında yakınsak hem de sınırlı, regüler yakınsak ve mutlak toplanabilir olan dizilerin $\widetilde{\mathcal{M}}_u(t)$, $\widetilde{\mathcal{C}}_p(t), \widetilde{\mathcal{C}}_{0p}(t)$, $\widetilde{\mathcal{C}}_{bp}(t)$, $\widetilde{\mathcal{C}}_r(t)$ ve $\widetilde{\mathcal{L}}_u(t)$ paranormlu dizi uzaylarını tanımlıyoruz. Ayrıca, $\widetilde{\mathcal{M}}_u(t)$, $\widetilde{\mathcal{C}}_p(t), \widetilde{\mathcal{C}}_{0p}(t)$, $\widetilde{\mathcal{C}}_r(t)$, $\widetilde{\mathcal{C}}_{bp}(t)$ ve $\widetilde{\mathcal{L}}_u(t)$ double dizi uzaylarının bazı özelliklerini inceliyoruz.

Anahtar kelimeler. Çift dizi uzayları, birinci mertebeden çift indisli Cesáro matrisi, paranormlu çift dizi uzayları.

1 Preliminaries, background and notation

By ω and Ω, we denote the sets of all real-valued single and double sequences which are the vector spaces with coordinatewise addition and scalar multiplication, respectively. Any vector subspaces of ω and Ω are called as the *single sequence space* and *double sequence space*, respectively.

By $\mathcal{M}_u$, we denote the space of all bounded double sequences, that is,

$$\mathcal{M}_u := \left\{x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty\right\}$$

which is a Banach space with the norm $\|x\|_\infty$, where $\mathbb{N}$ denotes the set of all positive integers.

Consider a sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{R}$ such that $|x_{mn} - l| < \varepsilon$ for all $m, n > n_0$, then we call that the double sequence x is *convergent* in the *Pringsheim's sense* to the limit l and write $p - \lim x_{mn} = l$, where $\mathbb{R}$ denotes the real field. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. Indeed, following Boos [1], if we define the sequence $x = (x_{mn})$ by

$$x_{mn} := \begin{cases} n, & m = 1, n \in \mathbb{N}, \\ 0, & m \geq 2, n \in \mathbb{N}, \end{cases}$$

then it is trivial that $x \in \mathcal{C}_p \backslash \mathcal{M}_u$, since $p - \lim x_{mn} = 0$ but $\|x\|_\infty = \infty$. So, we may mention the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. Following Hardy [6], a double sequence $x = (x_{mn})$ is said to *converge regularly* if it converges in Prinsheim's sense and, in addition, the following finite limits exist:

$$\begin{aligned} \lim_{n\to\infty} x_{mn} &= k_m \quad (m = 1, 2, ...), \\ \lim_{m\to\infty} x_{mn} &= l_n \quad (n = 1, 2, ...). \end{aligned}$$

For more details, see also [5]. Obviously, the regular convergence of x implies the convergence in Prinsheim's sense as well as the boundedness of the terms of x, but the converse implication fails. Also by $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$, we denote the spaces of all double sequences converging to 0 contained in the sequence spaces $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively. Móricz [7] proved that $\mathcal{C}_{bp}, \mathcal{C}_{bp0}, \mathcal{C}_r$ and $\mathcal{C}_{r0}$ are Banach spaces with the norm $\|.\|_\infty$.

Let us consider a double sequence $x = (x_{mn})$ and define the sequence $s = (s_{mn})$ which will be used throughout via x by

$$s_{mn} := \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} \tag{1.1}$$

for all $m, n \in \mathbb{N}$. For the sake of brevity, here and in what follows, we abbreviate the summation $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}$ by $\sum_{i,j}$ and we use this abbreviation with other letters. Let λ be a space of a double sequences converging with respect to some linear convergence rule $v-\lim : \lambda \to \mathbb{R}$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $v-\sum_{i,j} x_{ij} = v-\lim_{m,n\to\infty} s_{mn}$. Let λ, μ be two spaces of double sequences converging with respect to the linear convergence rules $v_1 - \lim$ and $v_2 - \lim$, respectively, and let $A = (a_{mnkl})$ be a four dimensional infinite matrix over the real or complex field.

The $v-$*summability domain* $\lambda_A^{(v)}$ of the four dimensional infinite matrix $A = (a_{mnkl})$ in the space λ of a double sequences is defined by

$$\lambda_A^{(v)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \left(v - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n\in\mathbb{N}} \text{ exists and is in } \lambda \right\}. \tag{1.2}$$

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Now, following Zeltser [9], we note the terminology for double sequence spaces. A locally convex double sequence space λ is called a $DK-$*space*, if all of the seminorms $r_{kl} : \lambda \to \mathbb{R}$, $x = (x_{kl}) \mapsto |x_{kl}|$ for all $k, l \in \mathbb{N}$ are continuous. A $DK-$space with a Fréchet topology is called an $FDK-$*space*. A normed $FDK-$space is called a $BDK-$*space*. We record that C_r endowed with the norm $\|.\|_\infty : \mathcal{C}_r \to \mathbb{R}$, $x = (x_{kl}) \mapsto \sup_{k,l\in\mathbb{N}} |x_{kl}|$ is a $BDK-$space.

Let us define the following sets of double sequences:

$$\begin{aligned}
\mathcal{M}_u(t) &:= \left\{ (x_{mn}) \in \Omega : \sup_{m,n\in\mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\}, \\
\mathcal{C}_p(t) &:= \left\{ (x_{mn}) \in \Omega : \exists l \in \mathbb{C} \ni p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 0 \right\}, \\
\mathcal{C}_{0p}(t) &:= \left\{ (x_{mn}) \in \Omega : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 0 \right\}, \\
\mathcal{L}_u(t) &:= \left\{ (x_{mn}) \in \Omega : \sum_{m,n} |x_{mn}|^{t_{mn}} < \infty \right\}, \\
\mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \quad \text{and} \quad \mathcal{C}_{0bp}(t) := \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t),
\end{aligned}$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$. In the case when $t_{mn} = 1$ for all $m, n \in \mathbb{N}$, $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively.

Now, we can summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [10–12] have proved that $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{L}_u(t)$ are complete paranormed spaces of double sequences and given the alpha-, beta-, gamma-duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, Zeltser [8] essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [13] introduced the statistical convergence and statistical Cauchy for double sequences, and gave the relation between statistically convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [14] and Mursaleen and Edely [15] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into the one whose core is a subset of the $M-$core of x. Altay and Başar [16] defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the alpha-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)-$duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. More recently, Başar and Sever [17] introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of absolutely $q-$summable single sequences and examine some properties of the space $\mathcal{L}_q$.

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. Robinson [19] presented a four dimensional analogue of regularity for double sequences in 1926. Accordingly, a four dimensional matrix A is said to be bounded-regular or RH-regular if it maps every bounded p-convergent sequence into a p-convergent sequence with the same p-limit, i.e, a matrix $A = (a_{mnkl})$ is said to be $RH-$regular if $Ax \in \mathcal{C}_{bp}$ and $p - \lim Ax = p - \lim x$ for each $x \in \mathcal{C}_{bp}$. The Cesàro matrix is well-known $RH-$regular matrix which is defined below.

The four-dimensional Cesàro matrix $C = (c_{mnkl})$ of order one is defined by

$$c_{mnkl} := \begin{cases} \dfrac{1}{(m+1)(n+1)} & , \quad 0 \leq k \leq m, \quad 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$. In this paper, we assume that the terms of the double sequences

$x = (x_{mn})$ and $y = (y_{mn})$ are connected with the relation

$$y_{mn} = (Cx)_{mn} = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij}$$

for all $m, n \in \mathbb{N}$. Quite recently, Mursaleen and Başar [18] defined the spaces $\widetilde{\mathcal{M}}_u$, $\widetilde{\mathcal{C}}_p$, $\widetilde{\mathcal{C}}_{0p}$, $\widetilde{\mathcal{C}}_{bp}$, $\widetilde{\mathcal{C}}_r$ and $\widetilde{\mathcal{L}}_q$ of double sequences whose Cesàro mean transforms are bounded, convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, regularly convergent and Cesàro absolutely $q-$summable, respectively, that is,

$$\begin{aligned}
\widetilde{\mathcal{M}}_u \quad &: \quad = \left\{ (x_{ij}) \in \Omega : \sup_{m,n\in\mathbb{N}} \frac{1}{(m+1)(n+1)} \left| \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} \right| < \infty \right\}, \\
\widetilde{\mathcal{C}}_p \quad &: \quad = \left\{ (x_{ij}) \in \Omega : \exists\, l \in \mathbb{C} \ni p - \lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} \left| \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} - l \right| = 0 \right\}, \\
\widetilde{\mathcal{C}}_{0p} \quad &: \quad = \left\{ (x_{ij}) \in \Omega : p - \lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} \left| \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} \right| = 0 \right\}, \\
\widetilde{\mathcal{L}}_q \quad &: \quad = \left\{ (x_{ij}) \in \Omega : \sum_{m,n} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} \right|^q < \infty \right\}.
\end{aligned}$$

Furthermore, they studied some topological properties of these spaces and characterized some matrix classes.

In the present paper, we introduce the new double paranormed Cesàro sequence spaces $\widetilde{\mathcal{M}}_u(t)$, $\widetilde{\mathcal{C}}_p(t)$, $\widetilde{\mathcal{C}}_{0p}(t)$ and $\widetilde{\mathcal{L}}_u(t)$, that is,

$$\begin{aligned}
\widetilde{\mathcal{M}}_u(t) \quad &: \quad = \left\{ (x_{mn}) \in \Omega : \sup_{m,n\in\mathbb{N}} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} \right|^{t_{mn}} < \infty \right\}, \\
\widetilde{\mathcal{C}}_p(t) \quad &: \quad = \left\{ (x_{mn}) \in \Omega : \exists\, l \in \mathbb{C} \ni p - \lim_{m,n\to\infty} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} - l \right|^{t_{mn}} = 0 \right\}, \\
\widetilde{\mathcal{C}}_{0p}(t) \quad &: \quad = \left\{ (x_{mn}) \in \Omega : p - \lim_{m,n\to\infty} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} \right|^{t_{mn}} = 0 \right\}, \\
\widetilde{\mathcal{L}}_u(t) \quad &: \quad = \left\{ (x_{ij}) \in \Omega : \sum_{m,n} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} \right|^{t_{mn}} < \infty \right\}.
\end{aligned}$$

Also, by $\widetilde{\mathcal{C}}_{bp}(t)$ and $\widetilde{\mathcal{C}}_r(t)$, we denote the sets of all the paranormed Cesàro convergent and bounded, and the paranormed Cesàro regularly convergent double sequences, respectively. When all terms of (t_{mn}) are constant and all are equal to $t > 0$, then we obtain $\widetilde{\mathcal{M}}_u(t) = \widetilde{\mathcal{M}}_u$, $\widetilde{\mathcal{L}}_u(t) = \widetilde{\mathcal{L}}_q$ and when all terms of (t_{mn}), excluding the first finite number of m and n, are constant and all are equal to $t > 0$, then we obtain $\widetilde{\mathcal{C}}_p(t) = \widetilde{\mathcal{C}}_p$ and $\widetilde{\mathcal{C}}_{0p}(t) = \widetilde{\mathcal{C}}_{0p}$, see [18]. One can easily see that the spaces $\widetilde{\mathcal{M}}_u(t)$, $\widetilde{\mathcal{C}}_p(t)$, $\widetilde{\mathcal{C}}_{0p}(t)$, $\widetilde{\mathcal{C}}_r(t)$, $\widetilde{\mathcal{C}}_{bp}(t)$ and $\widetilde{\mathcal{L}}_u(t)$ are the domain of the double Cesàro matrix C in the spaces $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{C}_r(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{L}_u(t)$, respectively.

2 Some new paranormed spaces of double sequences

In the present section, we give some topological and algebraical properties of the spaces $\widetilde{\mathcal{M}}_u(t)$, $\widetilde{\mathcal{C}}_p(t)$, $\widetilde{\mathcal{C}}_{0p}(t)$, $\widetilde{\mathcal{C}}_{bp}(t)$ and $\widetilde{\mathcal{L}}_u(t)$.

Firstly, let's give the following theorems which state under which conditions our spaces are linear space and complete paranormed space.

Theorem 2.1 *The space $\widetilde{\mathcal{M}}_u(t)$ is a linear space if and only if $H = \sup_{m,n} t_{mn} < \infty$.*

Proof. Suppose that $H < \infty$, $M = \max\{1, H\}$ and $x, y \in \widetilde{\mathcal{M}}_u(t)$. Then, there exist some K_x and K_y such that

$$\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}\right|^{t_{mn}} \leq K_x^M \quad \text{and} \quad \left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} y_{ij}\right|^{t_{mn}} \leq K_y^M$$

for all $m, n \in \mathbb{N}$. Since $t_{mn}/M < 1$ for all $m, n \in \mathbb{N}$, one can see that

$$\begin{aligned}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}(x_{ij}+y_{ij})\right|^{t_{mn}/M} &\leq \left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}\right|^{t_{mn}/M} \\ &\quad + \left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} y_{ij}\right|^{t_{mn}/M} \\ &\leq K_x + K_y\end{aligned}$$

which leads to $x + y \in \widetilde{\mathcal{M}}_u(t)$.

Now, suppose that $\lambda \in \mathbb{C}$ and $x \in \widetilde{\mathcal{M}}_u(t)$. Then since the inequality

$$|\lambda|^{t_{mn}} \leq \max\{1, |\lambda|^M\}$$

holds for all $m, n \in \mathbb{N}$, it is easily obtained that

$$\begin{aligned}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} \lambda x_{ij}\right|^{t_{mn}} &= |\lambda|^{t_{mn}}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}\right|^{t_{mn}} \\ &\leq \max\{1, |\lambda|^M\}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}\right|^{t_{mn}}\end{aligned}$$

which means that $\lambda x \in \widetilde{\mathcal{M}}_u(t)$.

Conversely, suppose that $\widetilde{\mathcal{M}}_u(t)$ be a linear space and $H = \sup_{m,n} t_{mn} = \infty$. Then, there exist the sequences $(m(i))$ and $(n(j))$, one of them is strictly increasing and the other one is non-decreasing, such that

$$t_{m(i),n(j)} > i + j \tag{2.1}$$

for all positive integers i, j. Now, we consider the sequence

$$x = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{2} b^{m(i)n(j)},$$

where $b^{mn} = (b_{ij}^{mn})_{ij}$ defined by

$$b_{ij}^{mn} = \begin{cases} (m+1)(n+1), & i = m, \quad j = n, \\ -(m+1)(n+1), & i = m+1, \quad j = n, \\ -(n+1)(m+1), & i = m, \quad j = n+1, \\ (m+1)(n+1), & i = m+1, \quad j = n+1, \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

for all $m, n \in \mathbb{N}$. Then, we have the sequence $y = (y_{mn})$, where

$$y_{mn} = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} = \begin{cases} \frac{1}{2}, & m = m(i), \quad n = n(j), \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. Therefore, it follows by (2.1) that

$$\begin{aligned} \sup_{m,n\in\mathbb{N}} |y_{mn}|^{t_{mn}} &= \sup_{i,j\in\mathbb{N}} |y_{m(i),n(j)}|^{t_{m(i),n(j)}} = \sup_{i,j\in\mathbb{N}} 2^{-t_{m(i),n(j)}} \\ &\leq \sup_{i,j\in\mathbb{N}} 2^{-i-j} \leq \frac{1}{2}. \end{aligned}$$

Hence, $x \in \widetilde{\mathcal{M}}_u(t)$ but it is clearly that $4x \notin \widetilde{\mathcal{M}}_u(t)$ which contradicts the fact that $\widetilde{\mathcal{M}}_u(t)$ is a linear space. So, H must be finite. ■

Theorem 2.2 *$\widetilde{\mathcal{C}}_{bp}(t)$ and $\widetilde{\mathcal{L}}_u(t)$ are linear spaces if and only if $H = \sup_{m,n} t_{mn} < \infty$.*

Proof. The proof of this theorem is similar to Theorem 2.1. So, we omit the detail. ■

Theorem 2.3 *The space $\widetilde{\mathcal{C}}_p(t)$ is a linear space if and only if*

$$T = \lim_{N\to\infty} \sup_{m,n\geq N} t_{mn} < \infty.$$

Proof. Let $x, y \in \widetilde{\mathcal{C}}_p(t)$ and $T < \infty$. Then there exist some complex numbers L_1, L_2 such that

$$p-\lim_{m,n\to\infty}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}-L_1\right|^{t_{mn}}=0,$$

and

$$p-\lim_{m,n\to\infty}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}y_{ij}-L_2\right|^{t_{mn}}=0.$$

Also there exists $\tau > 0$ such that $t_{mn} < \tau$ for all sufficiently large m, n. Then, we consider the inequality

$$\begin{aligned}&\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}(x_{ij}+y_{ij})-(L_1+L_2)\right|^{t_{mn}/\tau}\\ \leq\ &\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}-L_1\right|^{t_{mn}/\tau}+\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}y_{ij}-L_2\right|^{t_{mn}/\tau}.\end{aligned}$$

We have that $x+y \in \widetilde{\mathcal{C}}_p(t)$. Furthermore, let $\gamma \in \mathbb{C}$ and $x \in \widetilde{\mathcal{C}}_p(t)$. Then, by the inequality

$$\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}\gamma x_{ij}-\gamma L_1\right|^{t_{mn}}\leq\max\{1,|\gamma|^{\tau}\}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}-L_1\right|^{t_{mn}},$$

we obtain $\gamma x \in \widetilde{\mathcal{C}}_p(t)$.

Conversely, let $\widetilde{\mathcal{C}}_p(t)$ be a linear space and suppose that $T = \infty$. Then, there exists a strictly increasing sequence $(N(i,j))$ of positive integers such that $t_{m(i),n(j)} > i+j$ for all $i, j \in \mathbb{N}$, where $m(i), n(j) \geq N(i,j) > 1$. So, we consider the sequence

$$x=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{2}b^{m(i)n(j)}.$$

Then, it is clear that $x = (x_{mn}) \in \widetilde{\mathcal{C}}_p(t)$. But since

$$|y_{m(i),n(j)}|=\left|\frac{1}{(m(i)+1)(n(j)+1)}\sum_{i=0}^{m(i)}\sum_{j=0}^{n(j)}4x_{ij}\right|^{t_{m(i),n(j)}}=2^{t_{m(i),n(j)}}>2^{i+j},$$

then $4x \notin \widetilde{\mathcal{C}}_p(t)$. Hence $T < \infty$. ∎

Theorem 2.4 *The space $\widetilde{\mathcal{C}}_{0p}(t)$ is a linear space if and only if*

$$T=\lim_{N\to\infty}\sup_{m,n\geq N}t_{mn}<\infty.$$

Proof. The proof of this theorem is similar to Theorem 2.3. So, we omit the detail. ∎

Theorem 2.5 *Let* $H = \sup_{m,n} t_{mn}$*,* $M = \max\{1, H\}$*. Then, the space* $\widetilde{\mathcal{M}}_u(t)$ *is a complete paranormed space with* g *defined by*

$$g(x) = \sup_{m,n\in\mathbb{N}} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} \right|^{t_{mn}/M}$$

if and only if $h = \inf_{m,n} t_{mn} > 0$*.*

Proof. Let $\widetilde{\mathcal{M}}_u(t)$ be a paranormed space with the paranorm g and let $h = 0$. Consider the sequence $(x^l) = (x_{ij}) \subset \widetilde{\mathcal{M}}_u(t)$ $(l \in \mathbb{N})$ defined by

$$x_{ij} = \begin{cases} (m+1)(n+1) & , \quad i = 0, \quad j = 0, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

and the sequence $\gamma = (\gamma_l) = (1/(1+l))$ of scalars such that

$$\gamma_l \to 0 \quad \text{as} \quad l \to \infty \quad \text{and} \quad \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} = 1$$

for all $m, n \in \mathbb{N}$. Since

$$g(\gamma_l x^l) = \sup_{m,n\in\mathbb{N}} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} \gamma_l x_{ij}^l \right|^{t_{mn}/M} = \sup_{m,n\in\mathbb{N}} |\gamma_l|^{t_{mn}/M} = 1,$$

we obtain a contradiction with $g(\gamma_l x^l) \to 0$ as $l \to \infty$. Therefore, $h > 0$.

Conversely, let $h > 0$. It is trivial that $g(\theta) = 0$ and $g(-x) = g(x)$ for all $x \in \widetilde{\mathcal{M}}_u(t)$. Also, it is clear that $g(x+y) \le g(x) + g(y)$ for all $x, y \in \widetilde{\mathcal{M}}_u(t)$.

Moreover, let (x^l) be any sequence in $\widetilde{\mathcal{M}}_u(t)$ such that $g(x^l - x) \to 0$ $(l \to \infty)$ and let (γ_l) be any sequence of scalars such that $\gamma_l \to \gamma$ $(l \to \infty)$. Then, there exists a positive real number K such that $|\gamma_l| \le K$ for all $l \in \mathbb{N}$. Thus, we have

$$\begin{aligned} g(\gamma_l x^l - \gamma x) &= \sup_{m,n\in\mathbb{N}} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} (\gamma_l x_{ij}^l - \gamma x_{ij}) \right|^{t_{mn}/M} \\ &\le \sup_{m,n\in\mathbb{N}} \left(|\gamma_l| \left| \frac{1}{(m+1)(n+1)} \left(\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^l - \sum_{j=0}^{n} x_{ij} \right) \right| \right)^{t_{mn}/M} \\ &\quad + \sup_{m,n\in\mathbb{N}} \left(|\gamma_l - \gamma| \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} \right| \right)^{t_{mn}/M} \\ &\le K g(x^l - x) + |\gamma_l - \gamma|^{t_{mn}/M} g(x). \end{aligned}$$

Therefore, it is easy to see that $g(\gamma_l x^l - \gamma x) \to 0$ as $l \to \infty$, so $(\widetilde{\mathcal{M}}_u(t), g)$ is a paranormed space.

Now, we prove the completeness of the space $(\widetilde{\mathcal{M}}_u(t), g)$. Let (x^r) be any Cauchy sequence in the space $\widetilde{\mathcal{M}}_u(t)$. Then, for a given $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$g(x^r - x^s) = \sup_{m,n\in\mathbb{N}} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} (x_{ij}^r - x_{ij}^s) \right|^{t_{mn}/M} < \varepsilon$$

for all $r, s > N$. Hence, we have that

$$\begin{aligned} & \left| \frac{1}{(m+1)(n+1)} \left(\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^r - \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^s \right) \right|^{t_{mn}/M} \\ \leq \quad & \sup \left| \frac{1}{(m+1)(n+1)} \left(\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^r - \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^s \right) \right|^{t_{mn}/M} < \varepsilon \end{aligned}$$

which shows that $\left(\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^r \right)_{r\in\mathbb{N}}$ is a Cauchy sequence of complex numbers. Then, from completeness of $\mathbb{C}$, this sequence converges, say

$$\left| \frac{1}{(m+1)(n+1)} \left(\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^r - \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^s \right) \right|^{t_{mn}/M} \to 0 \quad \text{as} \quad r \to \infty.$$

We now show that $x \in \widetilde{\mathcal{M}}_u(t)$. Since (x^r) is a Cauchy sequence in the space $\widetilde{\mathcal{M}}_u(t)$, there exists a positive number K such that $g(x^r) < K$. Also, since

$$\begin{aligned} g(x) \quad &= \quad \sup_{m,n\in\mathbb{N}} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} \right|^{t_{mn}/M} \\ &\leq \quad g(x^r) + \sup_{m,n\in\mathbb{N}} \left| \frac{1}{(m+1)(n+1)} \left(\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} - \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^r \right) \right|^{t_{mn}/M}, \end{aligned}$$

then by passing to limit as $r \to \infty$, we have that

$$g(x) \leq K + \varepsilon$$

which leads to the fact that $x \in \widetilde{\mathcal{M}}_u(t)$. ■

Theorem 2.6 *Let*

$$N_1 = \min\{n_0 : \sup_{m,n\geq n_0} |y_{mn}|^{t_{mn}} < \infty\}, \; N_2 = \min\{n_0 : \sup_{m,n\geq n_0} t_{mn} < \infty\},$$

and $N = \max\{N_1, N_2\}$. *Then, the spaces* $\widetilde{\mathcal{C}}_p(t)$ *and* $\widetilde{\mathcal{C}}_{0p}(t)$ *are complete paranormed spaces with* g *defined by*

$$g(x) = \lim_{N\to\infty} \sup_{m,n\geq N} \left| \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} \right|^{t_{mn}/M}$$

if and only if $\mu > 0$, *where* $\mu = \lim_{N\to\infty} \inf_{m,n\geq N} t_{mn}$ *and* $M = \max\{1, \sup_{m,n\geq N} t_{mn}\}$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.5. So, we omit the detail. ■

Theorem 2.7 *The space* $\widetilde{\mathcal{L}}_u(t)$ *is a complete paranormed space with* g *defined by*

$$g(x)=\left[\sum_{m,n=0}^{\infty}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}\right|^{t_{mn}}\right]^{1/M},$$

where $M=\max\{1,H\}$ *and* $H=\sup_{m,n}t_{mn}<\infty$.

Proof. It is clear that $g(\theta)=0$, $g(-x)=g(x)$. Let $x,y\in\widetilde{\mathcal{L}}_u(t)$, then we have that

$$\begin{aligned} g(x) &\le \left[\sum_{m,n=0}^{\infty}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}\right|^{t_{mn}}\right]^{1/M} \\ &\quad +\left[\sum_{m,n=0}^{\infty}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}y_{ij}\right|^{t_{mn}}\right]^{1/M} \\ &\le g(x)+g(y). \end{aligned}$$

Furthermore, for any $\lambda\in\mathbb{C}$, we have $g(\lambda x)\le\max\{1,|\lambda|\}g(x)$. Hence, $(\lambda,x)\to\lambda x$ is continuous at $\lambda=0$, $x=\theta$, and that the function $x\to\lambda x$ is continuous at $x=\theta$ whenever λ is a fixed scalar. If $x\in\widetilde{\mathcal{L}}_u(t)$ is fixed, and $\varepsilon>0$, we can choose $M,N>0$ such that

$$\begin{aligned} R(x) &= \sum_{m=0}^{M}\sum_{n=N+1}^{\infty}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}\right|^{t_{mn}} \\ &\quad +\sum_{m=M+1}^{\infty}\sum_{n=0}^{N}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}\right|^{t_{mn}} \\ &\quad +\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}x_{ij}\right|^{t_{mn}} \\ &< \varepsilon/2. \end{aligned}$$

Therefore, $R(\lambda x)<\varepsilon/2$ since $|\lambda|<1$ and $\delta>0$, so that $|\lambda|<\delta$ gives

$$\sum_{m,n=0}^{M,N}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}\lambda x_{ij}\right|^{t_{mn}}<\varepsilon/2.$$

Thus, $|\lambda|<\min\{1,\delta\}$ implies that $g(\lambda x)<\varepsilon$. Hence, the function $\lambda\to\lambda x$ is continuous at $\lambda=0$. So, $\widetilde{\mathcal{L}}_u(t)$ is a paranormed space.

Now, we show that the space $(\widetilde{\mathcal{L}}_u(t),g)$ is complete. Let (x^r) be any Cauchy sequence in the space $\widetilde{\mathcal{L}}_u(t)$. Then, for a given $\varepsilon>0$, there exists a positive integer $N=N(\varepsilon)$ such that

$$
\begin{aligned}
g(x^r - x^s) &= \left[\sum_{m,n=0}^{\infty}\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}(x_{ij}^r - x_{ij}^s)\right|^{t_{mn}}\right]^{1/M} \\
&= \left[\sum_{m,n=0}^{\infty}|y_{mn}^r - y_{mn}^s|^{t_{mn}}\right]^{1/M} < \varepsilon \qquad (2.3)
\end{aligned}
$$

for all $r, s > N$. Thus, we have that

$$|y_{mn}^r - y_{mn}^s| \leq g(x^r - x^s) < \varepsilon$$

which shows that $y^r = (y_{mn}^r)_{r\in\mathbb{N}} = \left(\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}^r\right)_{r\in\mathbb{N}}$ is a Cauchy sequence of complex numbers for every fixed $m, n \in \mathbb{N}$. Then, from completeness of $\mathbb{C}$, this sequence converges, say (y_{mn}^r) is convergent to y_{mn}, namely,

$$\lim_{r\to\infty} y_{mn}^r = y_{mn} \qquad (2.4)$$

for all $m, n \in \mathbb{N}$. So we can define the sequence $y = (y_{mn})$. Furthermore, one can see by (2.3) that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|y_{mn}^r - y_{mn}^s|^{t_{mn}} < \varepsilon^M,$$

for $r, s > N$. If we pass to the limit as $s \to \infty$, then we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|y_{mn}^r - y_{mn}|^{t_{mn}} < \varepsilon^M,$$

for $r > N$ by (2.4). Thus, the last inequality leads to the fact that $g(x^r - x) \leq \varepsilon$ for $r > N$ which gives the required result. ■

Now, we give some inclusion theorems with related to our new double sequence spaces.

Theorem 2.8 *We have the following statements:*

(i) $\widetilde{\mathcal{M}}_u(t) \subset \widetilde{\mathcal{M}}_u$ *if and only if* $h = \inf_{m,n} t_{mn} > 0$.

(ii) $\widetilde{\mathcal{M}}_u \subset \widetilde{\mathcal{M}}_u(t)$ *if and only if* $H = \sup_{m,n} t_{mn} < \infty$.

(iii) $\widetilde{\mathcal{M}}_u = \widetilde{\mathcal{M}}_u(t)$ *if and only if* $0 < h \leq H < \infty$.

Proof. *(i)* Let $\widetilde{\mathcal{M}}_u(t) \subset \widetilde{\mathcal{M}}_u$ and let $h = 0$. Then, there exist the sequences $(m(i))$ and $(n(j))$ such that, one of them is strictly increasing and the other one is non-decreasing, and

$$t_{m(i),n(j)} < \frac{1}{i+1} \qquad (2.5)$$

for all positive integers i, j. Let us define the sequence

$$x = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} ib^{m_i n_j}.$$

Then, we have the sequence $y = (y_{mn})$, where

$$y_{mn} = \frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} = \begin{cases} i, & m = m(i), \quad n = n(j), \quad 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} \sup_{m,n\in\mathbb{N}} |y_{mn}|^{t_{mn}} &= \sup_{i,j\in\mathbb{N}} |y_{m(i),n(j)}|^{t_{m(i),n(j)}} = \sup_{i,j\in\mathbb{N}} i^{t_{m(i),n(j)}} \\ &\leq \sup_{i,j\in\mathbb{N}} i^{\frac{1}{1+i}} \leq 2. \end{aligned}$$

Therefore, $x \in \widetilde{\mathcal{M}}_u(t)$, but it is clear that $x \notin \widetilde{\mathcal{M}}_u$, which is a contradiction. Hence, $h > 0$.

Conversely, let $x \in \widetilde{\mathcal{M}}_u(t)$ and $h > 0$. Then, there exists a positive real number K_x such that

$$\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}\right|^{t_{mn}} \leq K_x$$

for all $m, n \in \mathbb{N}$. So, we have that

$$\left|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij}\right| \leq K_x^{1/t_{mn}} \leq \max\{1, K_x^{1/t}\}$$

which lead us to the consequence that $x \in \widetilde{\mathcal{M}}_u$.

(ii) Suppose that $\widetilde{\mathcal{M}}_u \subset \widetilde{\mathcal{M}}_u(t)$ and let $H = \infty$. Then there exist the sequences $(m(i))$ and $(n(j))$ such that, one of them is strictly increasing and the other one is nondecreasing, and

$$t_{m(i),n(j)} > i + j$$

for all $i, j \in \mathbb{N}$. In this case if we consider the sequence $x = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 2b^{m_i n_j}$ for which

$$y_{mn} = \frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} = \begin{cases} 2, & m = m(i), \quad n = n(j), \quad 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$, we obtain that $x \in \widetilde{\mathcal{M}}_u$. However since $|y_{mn}|^{t_{mn}} = 2^{t_{m(i),n(j)}} > 2^{i+j}$, $x \notin \widetilde{\mathcal{M}}_u(t)$, which is a contradiction, i.e., $H < 0$.

Conversely, let $x \in \widetilde{\mathcal{M}}_u$ and $H < 0$. Then, there exists a positive real number K_x such that $|y_{mn}| \leq K_x$ for all $m, n \in \mathbb{N}$. Hence we have that

$$|y_{mn}|^{t_{mn}} \leq K_x^{t_{mn}} \leq \max\{1, K_x^H\}.$$

Therefore, $x \in \widetilde{\mathcal{M}}_u(t)$.

(iii) The proof of *(iii)* follows from *(i)* and *(ii)*. ■

Lemma 2.1 *[18] $\mathcal{C}_p$ is the subspace of the space $\widetilde{\mathcal{C}}_p$.*

Theorem 2.9 *We have the following statements:*

(i) $\widetilde{\mathcal{C}}_p \subset \widetilde{\mathcal{C}}_p(t)$ *if and only if* $\mu = \lim_{N\to\infty} \inf_{m,n\geq N} t_{mn} > 0$.

(ii) $\widetilde{\mathcal{C}}_p(t) \subset \widetilde{\mathcal{C}}_p$ *if and only if* $T = \lim_{N\to\infty} \sup_{m,n\geq N} t_{mn} < \infty$.

(iii) $\widetilde{\mathcal{C}}_p(t) = \widetilde{\mathcal{C}}_p$ *if and only if* $0 < \mu \leq T < \infty$.

Proof. We only prove *(i)* because the proofs of *(ii)* and *(iii)* are the same of the proof of *(i)*. Suppose that $\mu > 0$ and $x \in \widetilde{\mathcal{C}}_p$. Then, there exists $L \in \mathbb{C}$ such that

$$\lim_{m,n} |y_{mn} - L| = 0.$$

Since $\mu > 0$, we may find a $\gamma > 0$ such that $t_{mn} > \gamma$ for all sufficiently large $m, n \in \mathbb{N}$. Moreover, since for every $0 < \varepsilon < 1$ and all sufficiently large $m, n \in \mathbb{N}$, $|y_{mn} - L| < \varepsilon^{1/\gamma}$, we have

$$|y_{mn} - L|^{t_{mn}} < |y_{mn} - L|^{\gamma} < \varepsilon,$$

that is, $x \in \widetilde{\mathcal{C}}_p(t)$.

Conversely, let $\widetilde{\mathcal{C}}_p \subset \widetilde{\mathcal{C}}_p(t)$ and suppose that $\mu = 0$. Then there exists a strictly increasing sequence $(N(i))$ of positive integers such that $t_{m(i),n(i)} < 1/i$ for all $i \in \mathbb{N}$, where $m(i), n(i) \geq N(i) > 1$. If we consider the sequence $x = (x_{mn})$ which is defined in [10, Section 2, Theorem 1], by

$$x_{mn} = \begin{cases} n^{1/t_{mn}}, & m = 1, \quad n = 1, 2, ..., \\ \left[\dfrac{z_{mn}+1}{4}\right]^{1/t_{mn}}, & m = m(i), \quad n = n(i), \\ 0, & \text{otherwise}, \end{cases}$$

where each z_{mn} is either 0 or 1 for every $m, n \in \mathbb{N}$. In [10], it is shown that $x \in \mathcal{C}_p$ but $x \notin \mathcal{C}_p(t)$. Furthermore, $\mathcal{C}_p(t) \subset \widetilde{\mathcal{C}}_p(t)$ since C is $RH-$regular. Hence, $x \notin \widetilde{\mathcal{C}}_p(t)$. Then, we obtain, by Lemma 2.1, $x \in \widetilde{\mathcal{C}}_p$ but $x \notin \widetilde{\mathcal{C}}_p(t)$. Hence, $\mu > 0$. ■

3 Conclusion

In [18], Mursalaen and Başar introduced the spaces $\widetilde{\mathcal{M}}_u$, $\widetilde{\mathcal{C}}_p$, $\widetilde{\mathcal{C}}_{0p}$, $\widetilde{\mathcal{C}}_{bp}$, $\widetilde{\mathcal{C}}_r$ and $\widetilde{\mathcal{L}}_q$ of double sequences whose Cesàro mean transforms are bounded, convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, regularly convergent and Cesàro absolutely $q-$summable, respectively. If we define the four dimensional matrix $C = (c_{mnkl})$ by

$$c_{mnkl} := \begin{cases} \dfrac{1}{(m+1)(n+1)} & , \quad 0 \leq k \leq m, \quad 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$, then one can consider the spaces $\widetilde{\mathcal{M}_u}$, $\widetilde{\mathcal{C}_p}$, $\widetilde{\mathcal{C}_{0p}}$, $\widetilde{\mathcal{C}_{bp}}$, $\widetilde{\mathcal{C}_r}$ and $\widetilde{\mathcal{L}_q}$ as the domain of the four dimensional matrix C in the spaces $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_1$, respectively.

In spite of the domain of certain triangle matrices in the normed or paranormed space of the single sequences are studied by several researches, the corresponding problems remain open for the four dimensional matrices and the double sequence spaces. As a natural continuation of [16,18], we have worked on the domain of Cesàro mean C of order one in the spaces $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{C}_{bp}(t)$, $\mathcal{C}_r(t)$ and $\mathcal{L}_q(t)$. As a beginning of the domain of a four dimensional matrix in some paranormed double sequence spaces, the main result of the present paper are meaningful and have an advantage among the papers related to the double sequence spaces.

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