Journal of Number Theory 133 (2013) 3119–3126

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A new method to compute the terms of generalized order-k Fibonacci numbers

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ARTICLE INFO

Article history:
Received 24 January 2013
Revised 18 March 2013
Accepted 18 March 2013
Available online 14 May 2013
Communicated by David Goss

MSC:
primary 11B37
secondary 15A15, 15A51

Keywords:
Fibonacci numbers
k sequences of the generalized order-k Fibonacci numbers
Hessenberg matrix

ABSTRACT

In this paper, we give a new method to determine the terms of generalized order-k Fibonacci numbers and Fibonacci type sequences by using the inverse of various Hessenberg and triangular matrices. In addition, instead of obtaining the n-th term only, we are able to determine successive $(n+1)$ terms of these sequences with this method simultaneously.

1. Introduction

There is quite an interest in the theory and applications of Fibonacci numbers and their generalizations. Miles [9] defined generalized order-k Fibonacci numbers as

$$f_{k,n} = \sum_{j=1}^{k} f_{k,n-j} \tag{1}$$

for $n > k \geqslant 2$, with boundary conditions: $f_{k,1} = f_{k,2} = f_{k,3} = \cdots = f_{k,k-2} = 0$ and $f_{k,k-1} = f_{k,k} = 1$.

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.03.007

Since calculating any term of these sequences by recurrence relation is very difficult, there is a need to find other methods. Many researchers obtained terms of Fibonacci numbers and generalized order-k Fibonacci numbers by using determinant of different Hessenberg matrices. For example, Öcal et al. [10] gave some determinantal and permanental representations of k-generalized Fibonacci numbers and obtained Binet's formulas for these sequences. Kılıç and Taşcı [5] studied on permanents and determinants of Hessenberg matrices. Yılmaz and Bozkurt [12] derived permanents and determinants of a type of Hessenberg matrices. Kaygısız and Şahin [2–4] gave some determinantal and permanental representations of Fibonacci type numbers. Li et al. [8] gave three new Fibonacci Hessenberg matrices and obtained determinants of them. In [7,6], authors defined two $(0,1)$-matrices and showed that the permanents of these matrices are the generalized Fibonacci and Lucas numbers.

Chen and Yu [1] presented three Hessenberg matrices

$$H=\begin{bmatrix} h_{11} & h_{12} & 0 & \cdots & 0 \\ h_{21} & h_{22} & h_{23} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & h_{n-1,n} \\ h_{n,1} & h_{n,2} & \cdots & h_{n,n-1} & h_{n,n} \end{bmatrix},$$

$$\widetilde{H}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ h_{11} & h_{12} & 0 & \ddots & \vdots & 0 \\ h_{21} & h_{22} & h_{23} & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & h_{n-1,n} & 0 \\ h_{n,1} & h_{n,2} & \cdots & h_{n,n-1} & h_{n,n} & 1 \end{bmatrix}$$

and

$$\widetilde{H}^{-1}=\left[\begin{array}{c|c} [\alpha]_{n\times 1} & [L]_{n\times n} \\ \hline h & [\beta^T]_{1\times n} \end{array}\right]_{(n+1)\times(n+1)}, \tag{2}$$

then they obtained equalities

$$\det(H)=(-1)^n h.\det(\widetilde{H}), \tag{3}$$

$$L=H^{-1}+h^{-1}\alpha\beta^T \tag{4}$$

and

$$H\alpha+he_n=0 \tag{5}$$

where e_n is n-th column of the identity matrix I_n.

The following lemma can be inferred from (3), (4) and (5).

Lemma 1.1. *Let e_1 be the first column of the identity matrix I_n and matrices L, β, h in (2). Then,*

$$\beta^T H+he_1=0. \tag{6}$$

Lemma 1.2. *(See [10].) Let $k \geqslant 2$ be an integer and $H_{n,k} = (h_{st})$ be an $n \times n$ Hessenberg matrix, where*

$$h_{st} = \begin{cases} i^{|s-t|}, & \text{if } -1 \leqslant s-t < k, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

then

$$\det(H_{n,k}) = f_{k,n+k-1} \tag{8}$$

where $i = \sqrt{-1}$.

Lemma 1.3. *(See [10].) Let $k \geqslant 2$ be an integer and $F_{n,k} = (f_{ij})$ be an $n \times n$ Hessenberg matrix, where*

$$f_{ij} = \begin{cases} -1, & \text{if } j = i+1, \\ 1, & \text{if } 0 \leqslant i-j < k, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

then

$$\det(F_{n,k}) = f_{k,n+k-1}. \tag{10}$$

Lemma 1.4. *(See [8].) Let $A_{n,t}$ be the $n \times n$ Hessenberg matrix in which the superdiagonal entries are 1, all main diagonal entries are 1 except the last one, which is $t+1$, and the entries of each diagonal below the main diagonal alternate 0's and 1's, starting with 0, and 1. Let $B_{n,t}$ be the matrix obtained from $A_{n,t}$ by alternately replacing the 1's on the superdiagonal with i's and $-i$'s. Then*

$$\det(A_{n,t}) = \det(B_{n,t}) = F_n + tF_{n-1}. \tag{11}$$

Lemma 1.5. *(See [11].) Let $E_n = (e_{ij})$ be an $n \times n$ Hessenberg matrix, where*

$$e_{ij} = \begin{cases} 3, & \text{if } j = i, \\ 1, & \text{if } |i-j| = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

then

$$\det(E_n) = F_{2n+2}. \tag{13}$$

The main purpose of this paper is to calculate successive $(n+1)$ terms of ordinary Fibonacci numbers and generalized order-k Fibonacci numbers by using Eqs. (3), (4), (5), (6) and matrices (7), (9), (12) and the matrix described in Lemma 1.4.

2. Main result

Now we give two theorems which give not only the m-th term of generalized order-k Fibonacci numbers $(f_{k,m})$ but also successive n terms of generalized order-k Fibonacci numbers $(f_{k,m-1}, \ldots, f_{k,m-n})$.

Theorem 2.1. *Let $k \geqslant 2$ be an integer, $H_{n,k}$ be an $n \times n$ Hessenberg matrix in (7) and*

$$\widetilde{H}_{n,k} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & & \vdots \\ & H_{n,k} & & 0 \\ & & & 1 \end{bmatrix}.$$

Then,

$$(\widetilde{H}_{n,k})^{-1} = \begin{bmatrix} f_{k,k-1} & 0 & 0 & \cdots & 0 & 0 \\ if_{k,k} & -if_{k,k-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ i^{(n-3)} f_{k,n+k-4} & i^{(n-1)} f_{k,n+k-5} & \ddots & \ddots & 0 & 0 \\ i^{(n-2)} f_{k,n+k-3} & i^{(n)} f_{k,n+k-4} & i^{(n-4)} f_{k,n+k-5} & \ddots & 0 & 0 \\ i^{(n-1)} f_{k,n+k-2} & i^{(n+1)} f_{k,n+k-3} & i^{(n-3)} f_{k,n+k-4} & \ddots & -if_{k,k-1} & 0 \\ i^{(n+1)} f_{k,n+k-1} & i^{(n-1)} f_{k,n+k-2} & i^{(n-2)} f_{k,n+k-3} & \cdots & if_{k,k} & f_{k,k-1} \end{bmatrix}$$

where $i = \sqrt{-1}$.

Proof. Let us construct the matrix

$$(\widetilde{H}_{n,k})^{-1} = \left[\begin{array}{c|c} [\alpha]_{n\times 1} & [L]_{n\times n} \\ \hline h & [\beta^T]_{1\times n} \end{array}\right]_{(n+1)\times(n+1)}$$

and obtain the entries;

(i) by using (3) and (8) we obtain

$$\det(H_{n,k}) = (-1)^n h.\det(\widetilde{H}_{n,k})$$
$$\Rightarrow \quad h = \frac{\det(H_{n,k})}{(-1)^n \det(\widetilde{H}_{n,k})} = \frac{f_{k,n+k-1}}{(-1)^n i^{n-1}} = i^{n+1} f_{k,n+k-1},$$

(ii) by using (5) and (6) we obtain

$$[\alpha] = -(H_{n,k})^{-1}(i)^{n+1} f_{k,n+k-1} e_n = \begin{bmatrix} f_{k,k-1} \\ if_{k,k} \\ \vdots \\ i^{(n-2)} f_{k,n+k-3} \\ i^{(n-1)} f_{k,n+k-2} \end{bmatrix}$$

and

$$[\beta^T] = -(H_{n,k})^{-1}(i)^{n+1} f_{k,n+k-1} e_1$$
$$\Rightarrow \quad [\beta] = \begin{bmatrix} i^{(n-1)} f_{k,n+k-2} & i^{(n-2)} f_{k,n+k-3} & \cdots & if_{k,k} & f_{k,k-1} \end{bmatrix},$$

(iii) by using (4) we obtain

$$[L] = (H_{n,k})^{-1} + \left((i)^{n+1} f_{k,n+k-1}\right)^{-1} \alpha \beta^T$$
$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -if_{k,k-1} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ i^{(n-1)} f_{k,n+k-5} & \ddots & \ddots & 0 & 0 \\ i^{(n)} f_{k,n+k-4} & i^{(n-4)} f_{k,n+k-5} & \ddots & 0 & 0 \\ i^{(n+1)} f_{k,n+k-3} & i^{(n-3)} f_{k,n+k-4} & \ddots & -if_{k,k-1} & 0 \end{bmatrix}.$$

Consequently, combining these three steps, we obtain the required result. □

Example 2.2. We obtain generalized order-4 Fibonacci numbers, by using Theorem 2.1;

$$\widetilde{H}_{5,4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & i & 0 & 0 & 0 & 0 \\ i & 1 & i & 0 & 0 & 0 \\ -1 & i & 1 & i & 0 & 0 \\ -i & -1 & i & 1 & i & 0 \\ 0 & -i & -1 & i & 1 & 1 \end{bmatrix}$$

and the inverse of $\widetilde{H}_{5,4}$ is

$$\widetilde{H}_{5,4}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ i & -i & 0 & 0 & 0 & 0 \\ -2 & 1 & -i & 0 & 0 & 0 \\ -4i & 2i & 1 & -i & 0 & 0 \\ 8 & -4 & 2i & 1 & -i & 0 \\ -15 & 8 & -4i & -2 & i & 1 \end{bmatrix}$$
$$= \begin{bmatrix} f_{4,3} & 0 & 0 & 0 & 0 & 0 \\ if_{4,4} & -if_{4,3} & 0 & 0 & 0 & 0 \\ -f_{4,5} & f_{4,4} & -if_{4,3} & 0 & 0 & 0 \\ -if_{4,6} & if_{4,5} & f_{4,4} & -if_{4,3} & 0 & 0 \\ f_{4,7} & -f_{4,6} & if_{4,5} & f_{4,4} & -if_{4,3} & 0 \\ -f_{4,8} & f_{4,7} & -if_{4,6} & -f_{4,5} & if_{4,4} & f_{4,3} \end{bmatrix}$$

which is the desired result.

Theorem 2.3. *Let $k \geqslant 2$ be an integer, $F_{n,k}$ be an $n \times n$ Hessenberg matrix in (9) and*

$$\widetilde{F}_{n,k} = \begin{bmatrix} 1 & 0 \;\cdots & 0 \\ & & \vdots \\ & F_{n,k} & 0 \\ & & 1 \end{bmatrix}.$$

Then,

$$(\widetilde{F}_{n,k})^{-1} = \begin{bmatrix} f_{k,k-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{k,k} & -f_{k,k-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ f_{k,n+k-4} & -f_{k,n+k-5} & \ddots & \ddots & 0 & 0 \\ f_{k,n+k-3} & -f_{k,n+k-4} & -f_{k,n+k-5} & \ddots & 0 & 0 \\ f_{k,n+k-2} & -f_{k,n+k-3} & -f_{k,n+k-4} & \ddots & -f_{k,k-1} & 0 \\ -f_{k,n+k-1} & f_{k,n+k-2} & f_{k,n+k-3} & \cdots & f_{k,k} & f_{k,k-1} \end{bmatrix}.$$

Proof. Proof is similar to the proof of Theorem 2.1, by using (10). □

Example 2.4. We obtain generalized order-4 Fibonacci numbers, by using Theorem 2.3;

$$\widetilde{F}_{5,4} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and the inverse of $\widetilde{F}_{5,4}$ is

$$\widetilde{F}_{5,4}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 & 0 \\ 4 & -2 & -1 & -1 & 0 & 0 \\ 8 & -4 & -2 & -1 & -1 & 0 \\ -15 & 8 & 4 & 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} f_{4,3} & 0 & 0 & 0 & 0 & 0 \\ f_{4,4} & -f_{4,3} & 0 & 0 & 0 & 0 \\ f_{4,5} & -f_{4,4} & -if_{4,3} & 0 & 0 & 0 \\ f_{4,6} & -f_{4,5} & -f_{4,4} & -f_{4,3} & 0 & 0 \\ f_{4,7} & -f_{4,6} & -f_{4,5} & -f_{4,4} & -f_{4,3} & 0 \\ -f_{4,8} & f_{4,7} & f_{4,6} & f_{4,5} & f_{4,4} & f_{4,3} \end{bmatrix}.$$

Now we give a new method to calculate the terms of Fibonacci type sequences that are obtained by using determinants of Hessenberg matrices.

Theorem 2.5. *Let A_n be an $n \times n$ Hessenberg matrix, K_n be a Fibonacci type sequence obtained by determinant of A_n, i.e.,* $\det A_n = K_n$ *and*

$$\widetilde{A}_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ & & & \vdots \\ & A_n & & 0 \\ & & & 1 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then,

$$(\widetilde{A}_n)^{-1} = \left[\begin{array}{c|c} [\alpha]_{n\times 1} & [L]_{n\times n} \\ \hline \frac{K_n}{(-1)^n \det \widetilde{A}_n} & [\beta]_{1\times n} \end{array}\right].$$

Proof. Proof is obvious from (3). □

Corollary 2.6. *Let $A_{n,t}$ be the matrix described in Lemma 1.4, F_n be the ordinary Fibonacci numbers and*

$$\widetilde{A}_{n,t} = \begin{bmatrix} 1 & 0 & 0 \\ & A_{n,t} & 0 \\ & & 1 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then,

$$(\widetilde{A}_{n,t})^{-1} = \begin{bmatrix} F_1 & 0 & \cdots & 0 & 0 \\ -F_2 & F_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{(n-2)} F_{n-2} & (-1)^{(n-3)} F_{n-3} & \ddots & 0 & 0 \\ (-1)^{(n-1)} F_{n-1} & (-1)^{(n-2)} F_{n-2} & \ddots & F_1 & 0 \\ (-1)^n (F_n + tF_{n-1}) & (-1)^{(n-1)} F_{n-1} & \cdots & -F_2 & F_1 \end{bmatrix}.$$

Proof. Proof is similar to the proof of Theorem 2.1, by using (11). □

Corollary 2.7. *Let E_n be the matrix in (12), F_n be an ordinary Fibonacci numbers and*

$$\widetilde{E}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ & & & \vdots \\ & E_n & & 0 \\ & & & 1 \end{bmatrix}_{(n+1)\times(n+1)}.$$

Then,

$$(\widetilde{E}_n)^{-1} = \begin{bmatrix} F_2 & 0 & \cdots & 0 & 0 \\ -F_4 & F_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{(n-2)} F_{2n-2} & (-1)^{(n-3)} F_{2n-4} & \ddots & 0 & 0 \\ (-1)^{(n-1)} F_{2n} & (-1)^{(n-2)} F_{2n-2} & \ddots & F_2 & 0 \\ (-1)^n F_{2n+2} & (-1)^{(n-1)} F_{2n} & \cdots & -F_4 & F_2 \end{bmatrix}.$$

Proof. Proof is similar to the proof of Theorem 2.1, by using (13). □

Conclusion 2.8. Many researchers obtained the terms of ordinary and generalized order-k Fibonacci numbers by using determinants and permanents of different Hessenberg matrices, see papers [3,5–8,10,12]. We gave a new method that can be applied to almost all of such Hessenberg matrices. In addition, other calculation methods give one term of these sequences in each calculation, while our method gives $(n+1)$ successive terms simultaneously by using the same matrices.

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