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SOME TOPOLOGICAL AND GEOMETRICAL PROPERTIES OF THE SEQUENCE SPACE $e^r(u,p)$*

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Abstract In this paper, we introduce the sequence space $e^r(u,p)$ and investigate its some topological and geometrical properties such as basis, α-, β-, γ- duals and the uniform Opial property.

Key words paranormed sequence spaces; matrix transformations; Luxemburg norm and uniform Opial property

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1 Introduction

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Let (λ, g) be a paranormed space. Recall that a sequence (b_k) of the elements of λ is called a basis for λ if and only if, for each $x \in \lambda$, there exists a unique sequence (α_k) of scalars such that

$$g\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) \to 0 \quad \text{as} \quad n \to \infty.$$

The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_k), written as $x = \sum \alpha_k b_k$.

By a sequence space, we mean any vector subspace of ω, the space of all real or complex valued sequences $x = (x_k)$. The well-known sequence spaces that we shall use throughout this paper are as follows:

ℓ_∞: the space of all bounded sequences,

c: the space of all convergent sequences,

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c_0: the space of all null sequences,

bs: the space of all sequences which form bounded series,

cs: the space of all sequences which form convergent series,

ℓ_1: the space of all sequences which form absolutely convergent series,

ℓ_p: the space of all sequences which form p-absolutely convergent series,

where $1 < p < \infty$.

Let $u = (u_k)$ be a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N} = \{0, 1, 2, \cdots\}$ and (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and also let $M = \max\{1, H\}$. Then, the linear spaces $\ell(p)$ and $\ell_\infty(p)$ were defined by Maddox [19] (see also Simons [27] and Nakano [23]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty \right\} \quad (0 < p_k \leq H < \infty)$$

and

$$\ell_\infty(p) = \left\{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty \right\},$$

which are the complete spaces paranormed by

$$g_1(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M} \quad \text{and} \quad g_2(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/M} \quad \Longleftrightarrow \quad \inf p_k > 0,$$

respectively.

The studies on the sequence spaces form a great part of the summability theory. For such type studies and more information we refer [1–8, 10, 15, 17, 21, 22, 26] and the others.

2 Preliminaries, Background and Notations

Let λ, μ be any two sequence spaces, $A = (a_{nk})$ be an infinite matrix of real or complex numbers and $x = (x_k)$ a sequence such that $Ax = (A_n(x)) = \left(\sum_k a_{nk} x_k\right)$ exists for each n. Then, the sequence $Ax = (A_n(x))$ is called the A-transform of x. We say that the matrix A maps λ into μ if Ax exists and belongs to μ for each $x \in \lambda$. By (λ, μ) we denote the set of all matrices which map λ into μ, where $n, k \in \mathbb{N}$. A sequence x is said to be A-summable to α if Ax converges to α, called the A-limit of x.

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We shall assume throughout that $0 < r < 1$, $p_k^{-1} + (p_k')^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. Also, we write $e^{(k)}$ to denote the sequence whose only non-zero term is 1 in k-th place for each $k \in \mathbb{N}$.

For the sake of brevity in notation, we write throughout that

$$\tilde{a}_{nk} = \sum_{j=k}^{\infty} \frac{\binom{j}{k} (r-1)^{j-k} r^{-j}}{u_k} a_{nj} \quad (n, k \in \mathbb{N}).$$

The Euler sequence spaces e_0^r and e_c^r were firstly studied by Altay and Başar [1]. The Euler sequence spaces were studied by many authors. For instance, see [1, 2, 10, 14, 22, 26].

Quite recently, Demiriz and Çakan [10] introduced paranormed Euler sequence spaces $e_0^r(u,p)$ and $e_c^r(u,p)$ and computed the α-, β- and γ- duals of these spaces.

For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in \omega : \quad Ax \in \lambda\}. \tag{2.1}$$

For example, the sequence space $\overline{\ell(p)}$ defined in [8] is the domain of the matrix $S = (s_{nk})$ in $\ell(p)$, where

$$s_{nk} = \begin{cases} 1, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. By the same technique the following sequence spaces were introduced in [3–6], respectively,

$$\overline{\ell(p)} = [\ell(p)]_S, \quad bs(p) = [\ell_\infty(p)]_S, \quad r^t(p) = [\ell(p)]_{R^t}, \quad r^t_\infty(p) = [\ell_\infty(p)]_{R^t},$$

where (t_k) is a sequence of positive numbers, $Q_n = \sum_{k=0}^{n} t_k$ and the matrix $R^t = (r^t_{nk})$ of the Riesz mean is defined by

$$r^t_{nk} = \begin{cases} \dfrac{t_k}{Q_n}, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

In this paper, we introduce the sequence space $e^r(u,p) = [\ell(p)]_{E^{r,u}}$ of non-absolute type, which is the set of all sequences whose $E^{r,u}$-transforms are in the spaces $\ell(p)$, where $E^{r,u}$ denotes the matrix $E^{r,u} = (e^r_{nk}(u))$ defined by

$$e^r_{nk}(u) = \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k u_k, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

Furthermore, we investigate some topological and geometrical properties of this new space, such as basis, the α-, β- and γ- duals and some results related to the uniform Opial property.

3 The Sequence Space $e^r(u,p)$ of Non-Absolute Type

Under the light of the above explanations, we can define the sequence space $e^r(u,p)$ as follows:

$$e^r(u,p) = \left\{ x = (x_j) \in \omega : \sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j u_j x_j \right|^{p_k} < \infty \right\}, \quad 0 < p_k \leq H < \infty.$$

Note that in the cases $(u_k) = e = (1, 1, \cdots)$ and $(u_k) = (p_k) = e = (1, 1, 1, \cdots)$, the sequence space $e^r(u,p)$ is reduced to the sequence spaces $e^r(p)$ and e^r_p which were introduced by Kara, Öztürk and Başarır [14] and Altay, Başar and Mursaleen [2], respectively.

Define the sequence $y = \{y_k(r)\}$, which will be frequently used, as the $E^{r,u}$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(r) = \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j u_j x_j \quad (k \in \mathbb{N}). \tag{3.1}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 3.1 $e^r(u,p)$ is the complete linear metric space paranormed by g, defined by

$$g(x)=\left[\sum_k\left|\sum_{j=0}^k\binom{k}{j}(1-r)^{k-j}r^ju_jx_j\right|^{p_k}\right]^{1/M},$$

where $0<p_k\le H<\infty$ for all $k\in\mathbb{N}$.

Proof The linearity of $e^r(u,p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x,z\in e^r(u,p)$ (see Maddox [18, p.30]):

$$\begin{aligned}&\left[\sum_k\left|\sum_{j=0}^k\binom{k}{j}(1-r)^{k-j}r^ju_j(x_j+z_j)\right|^{p_k}\right]^{1/M}\\ \le&\left[\sum_k\left|\sum_{j=0}^k\binom{k}{j}(1-r)^{k-j}r^ju_jx_j\right|^{p_k}\right]^{1/M}+\left[\sum_k\left|\sum_{j=0}^k\binom{k}{j}(1-r)^{k-j}r^ju_jz_j\right|^{p_k}\right]^{1/M}\end{aligned}\tag{3.2}$$

and for any $\alpha\in\mathbb{R}$ (see [18])

$$|\alpha|^{p_k}\le\max\{1,|\alpha|^M\}.\tag{3.3}$$

It is clear that $g(\theta)=0$ and $g(x)=g(-x)$ for all $x\in e^r(u,p)$. Again inequalities (3.2) and (3.3) yield the subadditivity of g and

$$g(\alpha x)\le\max\{1,|\alpha|\}g(x).\tag{3.4}$$

Let $\{x^n\}$ be any sequence of points $x^n\in e^r(u,p)$ such that $g(x^n-x)\to0$ and $\{\alpha_n\}$ also be any sequence of scalars such that $\alpha_n\to\alpha$. Then, since the inequality $g(x^n)\le g(x)+g(x^n-x)$ holds by the subadditivity of g, $\{g(x^n)\}$ is bounded. Thus, from inequality (3.4) and the subadditivity of g we thus have

$$\begin{aligned}g(\alpha_nx^n-\alpha x)&=\left[\sum_k\left|\sum_{j=0}^k\binom{k}{j}(1-r)^{k-j}r^ju_j(\alpha_nx_j^n-\alpha x_j)\right|^{p_k}\right]^{1/M}\\&=\left[\sum_k\left|\sum_{j=0}^k\binom{k}{j}(1-r)^{k-j}r^ju_j(\alpha_nx_j^n-\alpha x_j^n+\alpha x_j^n-\alpha x_j)\right|^{p_k}\right]^{1/M}\\&\le g(\alpha_nx^n-\alpha x^n)+g(\alpha x^n-\alpha x)\le|\alpha_n-\alpha|\quad g(x^n)+|\alpha|\quad g(x^n-x)\end{aligned}$$

which tends to zero as $n\to\infty$. This means that the scalar multiplication is continuous. Hence, g is a paranorm on the space $e^r(u,p)$.

It remains to prove the completeness of the space $e^r(u,p)$. Let $\{x^i\}$ be any Cauchy sequence in space $e^r(u,p)$, where $x^i=\{x_0^{(i)},x_1^{(i)},x_2^{(i)},\cdots\}$. Then, for a given $\varepsilon>0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^i-x^j)<\varepsilon\tag{3.5}$$

for all $i,j > n_0(\varepsilon)$. Using the definition of g, we obtain that, for each fixed $k \in \mathbb{N}$,

$$|(E^{r,u}x^i)_k - (E^{r,u}x^j)_k| \le \left[\sum_k \left|(E^{r,u}x^i)_k - (E^{r,u}x^j)_k\right|^{p_k}\right]^{1/M} < \varepsilon$$

for every $i,j \ge n_0(\varepsilon)$, which leads us to the fact that $\{(E^{r,u}x^0)_k, (E^{r,u}x^1)_k, (E^{r,u}x^2)_k, \cdots\}$ is a Cauchy sequence of real numbers for any fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(E^{r,u}x^i)_k \to (E^{r,u}x)_k$ as $i \to \infty$. Using these infinitely many limits $(E^{r,u}x)_0, (E^{r,u}x)_1, (E^{r,u}x)_2, \cdots$, we define the sequence $\{(E^{r,u}x)_0, (E^{r,u}x)_1, (E^{r,u}x)_2, \cdots\}$. From (3.5), for each $m \in \mathbb{N}$ and $i,j \ge n_0(\varepsilon)$, we have

$$\sum_{k=0}^{m} \left|(E^{r,u}x^i)_k - (E^{r,u}x^j)_k\right|^{p_k} \le g(x^i - x^j)^M < \varepsilon^M. \tag{3.6}$$

Take any $i \ge n_0(\varepsilon)$. First, let $j \to \infty$ in (3.6) after $m \to \infty$, to obtain $g(x^i - x) \le \varepsilon$. Finally, taking $\varepsilon = 1$ in (3.6) and letting $i \ge n_0(1)$, we have Minkowski's inequality for every fixed $m \in \mathbb{N}$:

$$\left[\sum_{k=0}^{m} |(E^{r,u}x)_k|^{p_k}\right]^{1/M} \le g(x^i - x) + g(x^i) \le 1 + g(x^i)$$

which implies that $x \in e^r(u,p)$. Since $g(x^i - x) \le \varepsilon$ for all $i \ge n_0(\varepsilon)$, it follows that $x^i \to x$ as $i \to \infty$, whence $e^r(u,p)$ is complete. □

Note that the absolute property does not hold on space $e^r(u,p)$, since there exists at least one sequence in the space $e^r(u,p)$ such that $g(x) \neq g(|x|)$; where $|x| = (|x_k|)$. This says that $e^r(u,p)$ is a sequence space of non-absolute type.

Theorem 3.2 The sequence space $e^r(u,p)$ of non-absolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$.

Proof To prove the theorem, we should show the existence of a linear bijection between the spaces $e^r(u,p)$ and $\ell(p)$. With the notation of (3.1), define the transformation T from $e^r(u,p)$ and $\ell(p)$ by $x \mapsto y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Now, let $y = (y_k) \in \ell(p)$ and define a sequence $x = \{x_k(r)\}$ by

$$x_k(r) = \sum_{j=0}^{k} \frac{\binom{k}{j}(r-1)^{k-j}r^{-k}}{u_k} y_j \quad (k \in \mathbb{N}).$$

Then, we have

$$\begin{aligned} g(x) &= \left[\sum_k \left|\sum_{j=0}^{k} \binom{k}{j}(1-r)^{k-j}r^j u_j x_j\right|^{p_k}\right]^{1/M} \\ &= \left(\sum_k \left|\sum_{j=0}^{k} \delta_{kj} y_j\right|^{p_k}\right)^{1/M} = \left(\sum_k |y_k|^{p_k}\right)^{1/M} = g_1(y) < \infty. \end{aligned}$$

Thus, we have that $x \in e^r(u,p)$ and consequently T is surjective. Hence, T is a linear bijection, which says that spaces $e^r(u,p)$ and $\ell(p)$ are linearly isomorphic, as desired. □

Let us suppose that $1 < p_k \le s_k$ for all $k \in \mathbb{N}$. Then, it is known that $\ell(p) \subset \ell(s)$ which leads us to the immediate consequence $e^r(u,p) \subset e^r(u,s)$.

Following theorem determines the basis for the paranormed space $e^r(u,p)$.

Theorem 3.3 Let $\lambda_k(r) = (E^{r,u}x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \le H < \infty$. Define the sequence $b^{(k)}(r) = \{b_n^{(k)}(r)\}_{n\in\mathbb{N}}$ of the elements of the space $e^r(u,p)$ by

$$b_n^{(k)}(r) = \begin{cases} 0, & 0 \le n < k, \\ \dfrac{1}{u_n}\dbinom{n}{k}(r-1)^{n-k}r^{-n}, & n \ge k \end{cases} \tag{3.7}$$

for every fixed $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)}(r)\}_{k\in\mathbb{N}}$ is a basis for the space $e^r(u,p)$ and any $x \in e^r(u,p)$ has a unique representation of the form

$$x = \sum_k \lambda_k(r) b^{(k)}(r). \tag{3.8}$$

Proof It is clear that $\{b^{(k)}(r)\}_{k\in\mathbb{N}} \subset e^r(u,p)$ since

$$E^{r,u}b^{(k)}(r) = e^{(k)} \in \ell(p) \qquad (k \in \mathbb{N}) \tag{3.9}$$

for $0 < p_k \le H < \infty$.

Let $x \in e^r(u,p)$ be given. For every non-negative integer m, put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) b^{(k)}(r). \tag{3.10}$$

Then, by applying $E^{r,u}$ to (3.10) under the fact (3.9), we obtain

$$E^{r,u}x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) E^{r,u} b^{(k)}(r) = \sum_{k=0}^{m} (E^{r,u}x)_k e^{(k)}$$

and

$$\left\{E^{r,u}(x - x^{[m]})\right\}_i = \begin{cases} 0, & 0 \le i \le m, \\ (E^{r,u}x)_i, & i > m, \end{cases}$$

where $i, m \in \mathbb{N}$. Given $\varepsilon > 0$, then there is an integer m_0 such that

$$\left[\sum_{i=m}^{\infty} |(E^{r,u}x)_i|^{p_k}\right]^{1/M} < \frac{\varepsilon}{2}$$

for all $m \ge m_0$. Hence,

$$g(x - x^{[m]}) = \left[\sum_{i=m}^{\infty} |(E^{r,u}x)_i|^{p_k}\right]^{1/M} \le \left[\sum_{i=m_0}^{\infty} |(E^{r,u}x)_i|^{p_k}\right]^{1/M} < \varepsilon$$

for all $m \ge m_0$, which proves that $x \in e^r(u,p)$ is represented as in (3.8).

Let us show the uniqueness of the representation for $x \in e^r(u,p)$ given by (3.8). Suppose, on the contrary, that there exists another representation $x = \sum_k \mu_k(r) b^{(k)}(r)$. Since the linear transformation T from $e^r(u,p)$ to $\ell(p)$, used in Theorem 3.2, is continuous, we have at this stage that

$$(E^{r,u}x)_n = \sum_k \mu_k(r)\{E^{r,u}b^{(k)}(r)\}_n = \sum_k \mu_k(r) e_n^{(k)} = \mu_n(r); \quad \text{for } n \in \mathbb{N}$$

which contradicts the fact that $(E^{r,u}x)_n = \lambda_n(r)$ for all $n \in \mathbb{N}$. $\square$

4 The α-, β- and γ- Duals of Space $e^r(u,p)$

In this section, we state and prove the theorems determining the α-, β- and γ- duals of the sequence space $e^r(u,p)$ of non-absolute type.

We shall first give the definition of α-, β- and γ- duals of a sequence space and later quote some lemmas which are needed in proving the theorems given in Section 4.

For the sequence spaces λ and μ, define a set $S(\lambda,\mu)$ by

$$S(\lambda,\mu)=\{z=(z_k):xz=(x_kz_k)\in\mu \ \text{ for all } \ x\in\lambda\}. \tag{4.1}$$

With the notation of (4.1), the α-, β- and γ- duals of a sequence space λ, which are respectively denoted by $\lambda^\alpha,\lambda^\beta$ and λ^γ, are defined by $\lambda^\alpha=S(\lambda,\ell_1)$, $\lambda^\beta=S(\lambda,cs)$ and $\lambda^\gamma=S(\lambda,bs)$.

Lemma 4.1 [12, Theorem 5.1.0 with $q_n=1$] (i) Let $1<p_k\le H<\infty$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):\ell_1)$ if and only if there exists an integer $B>1$ such that

$$\sup_{N\in\mathcal{F}}\sum_k\left|\sum_{n\in N}a_{nk}B^{-1}\right|^{p'_k}<\infty. \tag{4.2}$$

(ii) Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):\ell_1)$ if and only if

$$\sup_{N\in\mathcal{F}}\sup_{k\in\mathbb{N}}\left|\sum_{n\in N}a_{nk}\right|^{p_k}<\infty. \tag{4.3}$$

Lemma 4.2 [17, Theorem 1 (i)–(ii)] (i) Let $1<p_k\le H<\infty$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):\ell_\infty)$ if and only if there exists an integer $B>1$ such that

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}B^{-1}|^{p'_k}<\infty. \tag{4.4}$$

(ii) Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):\ell_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}}|a_{nk}|^{p_k}<\infty. \tag{4.5}$$

Lemma 4.3 [17, Corollary of Theorem 1] Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(p):c)$ if and only if (4.4) and (4.5) hold, and

$$\lim_{n\to\infty}a_{nk}=\beta_k, \quad k\in\mathbb{N}, \tag{4.6}$$

also holds.

Let $N\in\mathcal{F}$ and $N_k^*=N\cap\{n\in\mathbb{N}:n\ge k\}$. Define the sets follows:

$$e_1(p)=\left\{a=(a_k)\in\omega:\sup_{N\in\mathcal{F}}\sup_{k\in\mathbb{N}}\left|\sum_{n\in N_k^*}\frac{\binom{n}{k}(r-1)^{n-k}r^{-n}}{u_n}a_n\right|^{p_k}<\infty\right\},$$

$$e_2(p)=\bigcup_{B>1}\left\{a=(a_k)\in\omega:\sup_{N\in\mathcal{F}}\sum_k\left|\sum_{n\in N_k^*}\frac{\binom{n}{k}(r-1)^{n-k}r^{-n}}{u_n}a_nB^{-1}\right|^{p'_k}<\infty\right\},$$

$$e_3(p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{\binom{j}{k}(r-1)^{j-k} r^{-j} a_j}{u_k} B^{-1} \right|^{p'_k} < \infty \right\},$$

$$e_4(p) = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{j=k}^{n} \frac{\binom{j}{k}(r-1)^{j-k} r^{-j} a_j}{u_k} = l \quad \text{for some scalar} \quad l \right\},$$

$$e_5(p) = \left\{ a = (a_k) \in \omega : \sup_{n,k\in\mathbb{N}} \left| \sum_{j=k}^{n} \frac{\binom{j}{k}(r-1)^{j-k} r^{-j} a_j}{u_k} \right|^{p_k} < \infty \right\}.$$

Theorem 4.4 (i) Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then, $\{e^r(u,p)\}^\alpha = e_1(p)$.

(ii) Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $\{e^r(u,p)\}^\alpha = e_2(p)$.

Proof Since the case (i) may be proved by analogy, we give the proof only for case (ii). Let us take any $a = (a_k) \in \omega$. We easily derive from (3.1) that

$$a_n x_n = \sum_{k=0}^{n} \frac{\binom{n}{k}(r-1)^{n-k} r^{-n}}{u_n} a_n y_k = (T^r y)_n, \quad n \in \mathbb{N}, \tag{4.7}$$

where $T^r = (t^r_{nk})$ is defined by

$$t^r_{nk} = \begin{cases} \dfrac{\binom{n}{k}(r-1)^{n-k} r^{-n}}{u_n} a_n, & 0 \le k \le n, \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus, by combining (4.7) with Lemma 4.1 (i), we observe that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in e^r(u,p)$ if and only if $T^r y \in \ell_1$ whenever $y = (y_k) \in \ell(p)$. This means that $a = (a_n) \in \{e^r(u,p)\}^\alpha$ whenever $x = (x_n) \in e^r(u,p)$ if and only if $T^r \in (\ell(p) : \ell_1)$. This gives the result that $\{e^r(u,p)\}^\alpha = e_2(p)$. □

Theorem 4.5 (i) Let $0 < p_k \le 1$ for all $k \in \mathbb{N}$. Then, $\{e^r(u,p)\}^\beta = e_4(p) \cap e_5(p)$.

(ii) Let $1 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Then, $\{e^r(u,p)\}^\beta = e_3(p) \cap e_4(p)$.

Proof Since case (i) may be proved in a similar way, we prove only case (ii) as in the proof of Theorem 4.4. Consider the equation

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\frac{1}{u_k} \sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \right] a_k \\ &= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \frac{\binom{j}{k}(r-1)^{j-k} r^{-j} a_j}{u_k} \right] y_k = (B^r y)_n, \end{aligned} \tag{4.8}$$

where $B^r = (b^r_{nk})$ is defined by

$$b^r_{nk} = \begin{cases} \displaystyle\sum_{j=k}^{n} \frac{\binom{j}{k}(r-1)^{j-k} r^{-j} a_j}{u_k}, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

Thus, with (4.8), we deduce from Lemma 4.3 that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in e^r(u,p)$ if and only if $B^r y \in c$ whenever $y = (y_k) \in \ell(p)$. Therefore, we derive from (4.4) and

(4.6) that

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\frac{\binom{j}{k}(r-1)^{j-k}r^{-j}a_j}{u_k}B^{-1}\right|^{p_k'}<\infty$$

and

$$\lim_{n\to\infty}\sum_{j=k}^{n}\frac{\binom{j}{k}(r-1)^{j-k}r^{-j}a_j}{u_k}=\beta_k,$$

which shows that $\{e^r(u,p)\}^{\beta}=e_3(p)\cap e_4(p)$. □

Theorem 4.6 (i) Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $\{e^r(u,p)\}^{\gamma}=e_5(p)$.

(ii) Let $1<p_k\le H<\infty$ for all $k\in\mathbb{N}$. Then, $\{e^r(u,p)\}^{\gamma}=e_3(p)$.

Proof By (4.8) we see from Lemma 4.2 that $ax=(a_kx_k)\in bs$ whenever $x=(x_k)\in e^r(u,p)$ if and only if $B^ry\in\ell_\infty$ whenever $y=(y_k)\in\ell(p)$. Therefore, we respectively obtain from (4.5) and (4.4) that $\{e^r(u,p)\}^{\gamma}=e_5(p)$ for $0<p_k\le 1$, $\{e^r(u,p)\}^{\gamma}=e_3(p)$ for $1<p_k\le H<\infty$, which complete the proof. □

5 Matrix Mappings on the Sequence Space $e^r(u,p)$

In this section, we desire to characterize some matrix mappings on the space $e^r(u,p)$. Theorem 4.1 gives the exact conditions for the general case $0<p_k\le H<\infty$ by combining the cases $0<p_k\le 1$ and $1<p_k\le H<\infty$. We consider only the case $1<p_k\le H<\infty$ and leave the case $0<p_k\le 1$ to the reader because it may be proved in a similar fashion.

Theorem 5.1 (i) Let $1<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Then, $A\in(e^r(u,p):\ell_\infty)$ if and only if there exists an integer $B>1$ such that

$$E(B)=\sup_{n\in\mathbb{N}}\sum_k\left|\tilde{a}_{nk}B^{-1}\right|^{p_k'}<\infty,\tag{5.1}$$

$$\{a_{nk}\}_{k\in\mathbb{N}}\in e_3(p)\cap e_4(p),\quad n\in\mathbb{N}.\tag{5.2}$$

(ii) Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Then, $A\in(e^r(u,p):\ell_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}}|\tilde{a}_{nk}|^{p_k}<\infty,\tag{5.3}$$

$$\{a_{nk}\}_{k\in\mathbb{N}}\in e_4(p)\cap e_5(p),\quad n\in\mathbb{N}.\tag{5.4}$$

Proof (i) Suppose that conditions (5.1) and (5.2) hold, and $x\in e^r(u,p)$. In this situation, since $\{a_{nk}\}_{k\in\mathbb{N}}\in[e^r(u,p)]^{\beta}$ for every fixed $n\in\mathbb{N}$, the A-transform of x exists. Consider the following equality obtained by using relation (3.1) that

$$\sum_{k=0}^{m}a_{nk}x_k=\sum_{k=0}^{m}\sum_{j=k}^{m}\frac{\binom{j}{k}(r-1)^{j-k}r^{-j}a_{nj}}{u_k}y_k,\quad m,n\in\mathbb{N}.\tag{5.5}$$

Taking into account the hypothesis, we derive from (5.5) that, as $m\to\infty$,

$$\sum_k a_{nk}x_k=\sum_k\tilde{a}_{nk}y_k,\quad n\in\mathbb{N}.\tag{5.6}$$

Now, by combining (5.6) and the inequality which holds for any $B > 0$ and any complex numbers a, b

$$|ab| \leq B\left(\left|aB^{-1}\right|^{p'} + |b|^p\right), \tag{5.7}$$

where $p > 1$ and $p^{-1} + (p')^{-1}$ (see [17]), one can easily see that

$$\sup_{n\in\mathbb{N}}\left|\sum_k a_{nk}x_k\right| \leq \sup_{n\in\mathbb{N}}\sum_k |\tilde{a}_{nk}||y_k| \leq B\left[E(B) + g_1^B(y)\right] < \infty.$$

Conversely, suppose that $A \in (e^r(u,p) : \ell_\infty)$ and $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then Ax exists for every $x \in e^r(u,p)$ which implies that $\{a_{nk}\}_{k\in\mathbb{N}} \in [e^r(u,p)]^\beta$ for each $n \in \mathbb{N}$. Now, the necessity of (5.2) is immediate. Moreover, we have from (5.6) that the matrix $B = (b_{nk})$ defined by $b_{nk} = \tilde{a}_{nk}$ for all $n, k \in \mathbb{N}$, is the class $(\ell(p) : \ell_\infty)$. Then, B satisfies condition (4.4) which is equivalent to (5.1). □

Theorem 5.2 Let $0 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then, $A \in (e^r(u,p) : c)$ if and only if (5.1)–(5.4) hold, and there is a sequence (α_k) of scalars such that

$$\lim_{n\to\infty}\tilde{a}_{nk} = \alpha_k \quad (k \in \mathbb{N}). \tag{5.8}$$

Since the proof is routine we omit it.

Corollary 5.3 Let $0 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then, $A \in (e^r(u,p) : c_0)$ if and only if (5.1)–(5.4) hold, and (5.8) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.

Now, we may give our basic lemma which is useful for deriving the characterizations of certain matrix classes via Theorems 5.1 and 5.2 and Corollary 5.3.

Lemma 5.4 [22, Lemma 2.6] Let λ, μ be any two sequence spaces, A be an infinite matrix and B a triangular matrix. Then, $A \in (\lambda : \mu_B)$ if and only if $BA \in (\lambda : \mu)$.

It is trivial that Lemma 5.4 has several consequences. Indeed, combining Lemma 5.4 with Theorems 5.1 and 5.2 and Corollary 5.3, one can easily derive the following results.

Corollary 5.5 Let $A = (a_{nk})$ be an infinite matrix and $t = (t_k)$ be a sequence of positive numbers. Define the matrix $R = (r_{nk})$ by

$$r_{nk} = \frac{1}{T_n}\sum_{j=0}^{n} t_j a_{jk} \quad (n, k \in \mathbb{N}),$$

where $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for $A = (a_{nk})$ to belong to any of the classes $(e^r(u,p) : r_\infty^t)$, $(e^r(u,p) : r_c^t)$ or $(e^r(u,p) : r_0^t)$ are obtained from the respective ones in Theorems 5.1 and 5.2 and Corollary 5.3 by replacing the entries of the matrix A by those of the matrix R where the sequence spaces r_∞^t, r_c^t and r_0^t were defined by Malkowsky in [21].

Corollary 5.6 Let $A = (a_{nk})$ be an infinite matrix and $C = (c_{nk})$ and $D = (d_{nk})$ be defined respectively by $c_{nk} = a_{nk} - a_{n+1,k}$ and $d_{nk} = a_{nk} - a_{n-1,k}$ for all $k, n \in \mathbb{N}$. Then, the necessary and sufficient conditions in order for A to belong to any of the classes $(e^r(u,p) : \ell_\infty(\Delta))$, $(e^r(u,p) : c(\Delta))$ or $(e^r(u,p) : c_0(\Delta))$ are obtained from the respective ones in Theorems 5.1 and 5.2 and Corollary 5.3 by replacing the entries of the matrix A by those of the matrices C and D, where $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$ denote the difference spaces of all bounded, convergent, null sequences and were introduced by Kızmaz [15].

Corollary 5.7 Let $A = (a_{nk})$ be an infinite matrix and define a matrix $E = (e_{nk})$ by

$$e_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-s)^{n-j} s^j a_{jk} \quad (n, k \in \mathbb{N}).$$

Then, the necessary and sufficient conditions in order for A to belong to any of the classes $(e^r(u,p) : e^s_\infty)$, $(e^r(u,p) : e^s_c)$ or $(e^r(u,p) : e^s_0)$ are obtained from the respective ones in Theorems 5.1 and 5.2 and Corollary 5.3 by replacing the entries of the matrix A by those of the matrix E, where $0 < s < 1$, e^s_∞ and e^s_c, e^s_0 denote the spaces of all sequences whose Euler transforms of order s are in the spaces ℓ_∞ and c, c_0, respectively, which were studied by Altay, Başar and Mursaleen [2] and Altay and Başar [1].

6 Uniform Opial Property of the Sequence Space $e^r(u,p)$

In functional analysis, the Opial property is an abstract property of Banach spaces that plays an important role in the study of weak convergence of iterates of mappings of Banach spaces, and of the asymptotic behaviour of nonlinear semigroups. The Opial property is important because Banach spaces with this property have the weak fixed-point property [13]. For details, we may refer to [11, 20]. In this section, we characterize the uniform Opial property of the space $e^r(u,p)$ and emphasize some results related to the this concept.

Let X be a linear space. By X^*, we denote the set of all bounded linear functionals on X and by $S(X)$ and $B(X)$, we denote the unit sphere and unit ball of a Banach space X, respectively.

Let $\{x_n\}$ be a sequence in a normed space X. Then $\{x_n\}$ is said to be weakly convergent to $x_0 \in X$, written as $x_n \to x$ (weakly), if and only if

$$f(x_n) \to f(x) \quad \text{as} \quad n \to \infty \quad \text{for every} \quad f \in X^*.$$

We call x_0 the weak limit of (x_n) [18].

A Banach space X is said to have the Opial property if every sequence $\{x_n\}$ weakly convergent to x_0 satisfies $\liminf_{n\to\infty} \|x_n - x_0\| \le \liminf_{n\to\infty} \|x_n - x\|$ for every $x \in X$ (see [24]).

Opial proved in [24] that the sequence space ℓ_p $(1 < p < \infty)$ has this property but $L_p[0, 2\pi]$ $(p \neq 2, 1 < p < \infty)$ does not.

A Banach space X is said to have the uniform Opial property (see [25]) if, for every $\varepsilon > 0$, there exists $\tau > 0$ such that, for each weakly null sequence $\{x_n\} \subset S(X)$ and $x \in X$ with $\|x\| \ge \varepsilon$, $1 + \tau \le \liminf_{n\to\infty} \|x_n + x\|$.

Now we give some definitions and lemmas concerned with our considerations:

Let X be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called modular if

(i) $\varrho(x) = 0$ if and only if $x = \theta$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalars α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \le \varrho(x) + \varrho(y)$ for all $x, y \in X$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

A modular ϱ is called convex if

(iv) $\varrho(\alpha x + \beta y) \le \alpha\varrho(x) + \beta\varrho(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

For any modular ϱ on X, the space $X_\varrho = \{x \in X : \varrho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$ is called the modular space.

A sequence $\{x_n\}$ of elements of X_ϱ is called modular convergent to $x \in X_\varrho$ if there exists a constant $\lambda > 0$ such that $\varrho(\lambda(x_n - x)) \to 0$ as $n \to \infty$.

A modular ϱ is said to satisfy the δ_2-condition ($\varrho \in \delta_2$) if, for any $\epsilon > 0$, there exist constants $K \geq 2$ and $a > 0$ such that $\varrho(2x) \leq K\varrho(x) + \epsilon$ for all $x \in X_\varrho$ with $\varrho(x) \leq a$.

If ϱ satisfies the δ_2-condition for any $a > 0$ with $K \geq 2$ dependent on a, we say that ϱ satisfies the strong δ_2-condition ($\varrho \in \delta_2^s$).

Lemma 6.1 [9, Lemma 2.3] Convergences in norm and in modular are equivalent in X_ϱ if $\varrho \in \delta_2$.

Lemma 6.2 [9, Lemma 2.1] If $\varrho \in \delta_2^s$, then for any $L > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\varrho(u+v) - \varrho(u)| < \epsilon$$

whenever $u, v \in X_\varrho$ with $\varrho(u) \leq L$ and $\varrho(v) \leq \delta$.

Lemma 6.3 [9, Lemma 2.4] If $\varrho \in \delta_2^s$, then, for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x\| \geq 1 + \delta$ whenever $\varrho(x) \geq 1 + \epsilon$.

For $x \in e^r(u,p)$, we define

$$\varrho_p(x) = \sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j u_j x_j \right|^{p_k}.$$

If $p_k \geq 1$ for all $k \in \mathbb{N}$, by the convexity of the function $t \mapsto |t|^{p_k}$ for each $k \in \mathbb{N}$, we have that ϱ_p is convex modular on $e^r(u,p)$. We consider $e^r(u,p)$ equipped with the Luxemburg norm given by

$$\|x\| = \inf\left\{\alpha > 0 : \varrho_p\left(\frac{x}{\alpha}\right) \leq 1\right\}. \tag{6.1}$$

We establish some basic properties for the modular ϱ_p.

Lemma 6.4 The modular ϱ_p on $e^r(u,p)$ satisfies the following properties:

(i) If $0 < \alpha < 1$, then $\alpha^M \varrho_p(x/\alpha) \leq \varrho_p(x)$ and $\varrho_p(\alpha x) \leq \alpha\varrho_p(x)$;

(ii) If $\alpha > 1$, then $\varrho_p(x) \leq \alpha^M \varrho_p(x/\alpha)$;

(iii) If $\alpha \geq 1$, then $\varrho_p(x) \leq \alpha\varrho_p(x/\alpha)$.

Proof (i) We have

$$\begin{aligned}\varrho_p(x) = \sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j u_j x_j \right|^{p_k} &= \sum_k \left| \frac{\alpha \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j u_j x_j}{\alpha} \right|^{p_k} \\ &\geq \alpha^M \sum_k \left| \frac{\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j u_j x_j}{\alpha} \right|^{p_k} = \alpha^M \varrho_p(x/\alpha).\end{aligned}$$

Since ϱ_p is convex, it follows that $\varrho_p(\alpha x) \leq \alpha\varrho_p(x)$.

(ii) If $\alpha \geq 1$, then, from (i), we have

$$\left(\frac{1}{\alpha}\right)^M \varrho_p(x) = \left(\frac{1}{\alpha}\right)^M \varrho_p\left(\frac{x/\alpha}{1/\alpha}\right) \leq \varrho_p\left(\frac{x}{\alpha}\right)$$

and hence $\varrho_p(x) \leq \alpha^M \varrho_p(x/\alpha)$.

Property (iii) follows from the convexity of ϱ_p. □

Now, we give some relationships between the modular ϱ_p and Luxemburg norm on $e^r(u,p)$.

Lemma 6.5 For any $x \in e^r(u,p)$, we have

(i) If $\|x\| < 1$, then $\varrho_p(x) \leq \|x\|$;

(ii) If $\|x\| > 1$, then $\varrho_p(x) \geq \|x\|$;

(iii) $\|x\| = 1$ if and only if $\varrho_p(x) = 1$;

(iv) $\|x\| < 1$ if and only if $\varrho_p(x) < 1$;

(v) $\|x\| > 1$ if and only if $\varrho_p(x) > 1$;

(vi) If $0 < \alpha < 1$ and $\|x\| > \alpha$, then $\varrho_p(x) > \alpha^M$;

(vii) If $\alpha \geq 1$ and $\|x\| < \alpha$, then $\varrho_p(x) < \alpha^M$.

Proof (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$. By the definition of the Luxemburg norm, there exists $\alpha > 0$ such that $\|x\| + \varepsilon > \alpha$ and $\varrho_p(\frac{x}{\alpha}) \leq 1$. From Lemma 5.4 (i) and (iii), we have

$$\varrho_p(x) \leq \varrho_p\left((\|x\| + \varepsilon)\frac{x}{\alpha}\right) \leq (\|x\| + \varepsilon)\varrho_p\left(\frac{x}{\alpha}\right) \leq \|x\| + \varepsilon.$$

Since ε is arbitrary, we have (i).

(ii) Let $\varepsilon > 0$ be such that $0 < \varepsilon < (\|x\| - 1)/\|x\|$, then $1 < (1-\varepsilon)\|x\| < \|x\|$. Combining the definition of the Luxemburg norm by Lemma 5.4 (i), we have

$$1 < \varrho_p\left(\frac{x}{(1-\varepsilon)\|x\|}\right) \leq \frac{1}{(1-\varepsilon)\|x\|}\varrho_p(x),$$

so $(1-\varepsilon)\|x\| < \varrho_p(x)$ for all $\varepsilon \in (0, (\|x\| - 1)/\|x\|)$. This implies that $\|x\| \leq \varrho_p(x)$.

Since ϱ_p is continuous, (iii) and (iv) directly follow from Theorem 1.4 of [20].

(iv) follows from (i) and (iii).

(v) follows from (iii) and (iv).

(vi) and (vii) follow from Lemma 5.4 (i) and (ii). □

Proposition 6.6 The space $e^r(u,p)$ is a Banach space with respect to the Luxemburg norm defined by (6.1).

Proof We will show that every Cauchy sequence in $e^r(u,p)$ is convergent according to the Luxemburg norm. Let $(x^n) = (x_j^n)$ be a Cauchy sequence in $e^r(u,p)$ and $\varepsilon \in (0,1)$. Thus, there exists n_0 such that $\|x^n - x^m\| < \varepsilon^M$ for all $m, n \geq n_0$. By Lemma 6.5 (i) we have

$$\varrho_p(x^n - x^m) < \|x^n - x^m\| < \varepsilon^M \tag{6.2}$$

for all $m, n \geq n_0$. That is,

$$\sum_k \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j u_j (x_j^n - x_j^m)\right|^{p_k} < \varepsilon^M.$$

For fixed j, we get that $|x_j^n - x_j^m| < \varepsilon$ for $m, n \geq n_0$. Thus, (x_j^n) is a Cauchy sequence in $\mathbb{R}$ for all $j \in \mathbb{N}$. Since $\mathbb{R}$ is complete for each $j \in \mathbb{N}$, $x_j^m \to x_j$ as $m \to \infty$. So, for fixed j, $|x_j^n - x_j| < \varepsilon$ for $n \geq n_0$. From inequality (6.2), we can write

$$\sum_k \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j u_j (x_j^n - x_j^m)\right|^{p_k} < \varepsilon \quad \text{for all } n, m \geq n_0.$$

Since $x_j^m \to x_j$ for every $j \in \mathbb{N}$, $\varrho_p(x^n - x^m) \to \varrho_p(x^n - x)$ as $m \to \infty$. So, we can write

$$\sum_k \left|\sum_{j=0}^k \binom{k}{j}(1-r)^{k-j} r^j u_j (x_j^n - x_j^m)\right|^{p_k} \to \sum_k \left|\sum_{j=0}^k \binom{k}{j}(1-r)^{k-j} r^j u_j (x_j^n - x_j)\right|^{p_k}$$

as $m \to \infty$. Hence, we have

$$\varrho_p(x^n - x) < \varepsilon \text{ for all } n \geq n_0.$$

From Lemma 6.5 (iv), we have $\|x^n - x\| < \varepsilon$. And now, by the linearity of the sequence space $e^r(u,p)$, we can write as follows:

$$x = (x - x^n) + x^n.$$

By the following calculations, we obtain that

$$\begin{aligned}
&\sum_k \left|\sum_{j=0}^k \binom{k}{j}(1-r)^{k-j} r^j u_j x_j\right|^{p_k} \\
&= \sum_k \left|\sum_{j=0}^k \binom{k}{j}(1-r)^{k-j} r^j u_j (x_j - x_j^n) + \sum_{j=0}^k \binom{k}{j}(1-r)^{k-j} r^j u_j x_j^n\right|^{p_k} \\
&\leq \sum_k \left|\sum_{j=0}^k \binom{k}{j}(1-r)^{k-j} r^j u_j (x_j - x_j^n)\right|^{p_k} + \sum_k \left|\sum_{j=0}^k \binom{k}{j}(1-r)^{k-j} r^j u_j x_j^n\right|^{p_k} \\
&\leq \varepsilon.
\end{aligned}$$

So, x is in $e^r(u,p)$. Hence, the sequence space $e^r(u,p)$ is a Banach space with respect to the Luxemburg norm. □

Lemma 6.7 Let $\limsup\limits_{k\to\infty} p_k < \infty$. Then, $e^r(u,p)$ equipped with the Luxemburg norm has the uniform Opial property.

Proof Take any $\varepsilon > 0$ and $x \in e^r(u,p)$ with $\|x\| \geq \varepsilon$. Let $\{x_n\}$ be a weakly null sequence in $S(e^r(u,p))$. Since $\limsup\limits_{k\to\infty} p_k < \infty$, we have $\varrho_p \in \delta_2^s$. Thus, by Lemma 6.1, there is $\mu \in (0,1)$ independent of x such that $\mu < \varrho_p(x) < \infty$. Also, since $\varrho_p \in \delta_2^s$, Lemma 6.2 asserts that there exists $\mu_1 \in (0,\mu)$ such that

$$|\varrho_p(y+z) - \varrho_p(y)| < \frac{\mu}{4} \tag{6.3}$$

whenever $\varrho_p(y) \leq 1$ and $\varrho_p(z) \leq \mu_1$. Since $\varrho_p(x) < \infty$, we can choose $m_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\sum_{m=m_0+1}^{\infty} \left|\sum_{j=m_0+1}^m \binom{m}{j}(1-r)^{m-j} r^j u_j x_j\right|^{p_m} &< \sum_{m=m_0+1}^{\infty} \left|\sum_{j=1}^m \binom{m}{j}(1-r)^{m-j} r^j u_j x_j\right|^{p_m} \\
&< \frac{\mu_1}{4}.
\end{aligned} \tag{6.4}$$

This gives

$$\begin{aligned}
\mu &< \sum_{m=1}^{m_0} \left|\sum_{j=1}^m \binom{m}{j}(1-r)^{m-j} r^j u_j x_j\right|^{p_m} + \sum_{m=m_0+1}^{\infty} \left|\sum_{j=1}^m \binom{m}{j}(1-r)^{m-j} r^j u_j x_j\right|^{p_m} \\
&\leq \sum_{m=1}^{m_0} \left|\sum_{j=1}^m \binom{m}{j}(1-r)^{m-j} r^j u_j x_j\right|^{p_m} + \frac{\mu_1}{4},
\end{aligned}$$

which implies

$$\sum_{m=1}^{m_0}\left|\sum_{j=1}^{m}\binom{m}{j}(1-r)^{m-j}r^j u_j x_j\right|^{p_m} > \mu - \frac{\mu_1}{4} > \mu - \frac{\mu}{4} = \frac{3\mu}{4}.$$

This together with the assumption $x_n \to 0$ (weakly), there exists $n_0 \in \mathbb{N}$ such that

$$\frac{3\mu}{4} \le \sum_{m=1}^{m_0}\left|\sum_{j=1}^{m}\binom{m}{j}(1-r)^{m-j}r^j u_j (x_n^{(j)} + x_j)\right|^{p_m} \tag{6.5}$$

for all $n > n_0$, since the weak convergence implies the coordinatewise convergence.

Again $x_n \to 0$ (weakly) implies there exists $n_1 > n_0$ such that $\varrho_p(x_{n|_{m_0}}) < \mu_1$ for all $n > n_1$. Since $\varrho_p(x_n) = 1$, by (6.3), we obtain

$$\left|\varrho_p(x_{n|_{\mathbb{N}-m_0}} + x_{n|_{m_0}}) - \varrho_p(x_{n|_{\mathbb{N}-m_0}})\right| < \frac{\mu}{4},$$

where $x_{|m} = (x^{(1)}, x^{(2)}, \cdots, x^{(m)}, 0, 0, \cdots)$ and $x_{|\mathbb{N}-m} = (0, 0, \cdots, x^{(m+1)}, x^{(m+2)}, \cdots)$. Hence,

$$1 - \frac{\mu}{4} = \varrho_p(x_n) - \frac{\mu}{4} < \varrho_p\left(x_{n|_{\mathbb{N}-m_0}}\right) = \sum_{m=m_0+1}^{\infty}\left|\sum_{j=1}^{m}\binom{m}{j}(1-r)^{m-j}r^j u_j (x_n^{(j)})\right|^{p_m}$$

for all $n > n_1$. This together with (6.3), (6.4) and (6.5) implies that, for any $n > n_1$,

$$\begin{aligned}
\varrho_p(x_n + x) &= \sum_{m=1}^{m_0}\left|\sum_{j=1}^{m}\binom{m}{j}(1-r)^{m-j}r^j u_j (x_n^{(j)} + x_j)\right|^{p_m} \\
&\quad + \sum_{m=m_0+1}^{\infty}\left|\sum_{j=1}^{m}\binom{m}{j}(1-r)^{m-j}r^j u_j (x_n^{(j)} + x_j)\right|^{p_m} \\
&\ge \frac{3\mu}{4} + \sum_{m=m_0+1}^{\infty}\left|\sum_{j=m_0+1}^{m}\binom{m}{j}(1-r)^{m-j}r^j u_j (x_n^{(j)})\right|^{p_m} - \frac{\mu}{4} \\
&\ge \frac{3\mu}{4} + (1 - \frac{\mu}{4}) - \frac{\mu}{4} = 1 + \frac{\mu}{4}.
\end{aligned}$$

Since $\varrho_p \in \delta_2^s$, by Lemma 6.3, there is a constant τ depending on μ only such that $\|x_n + x\| \ge 1 + \tau$. This means that $e^r(u, p)$ has the uniform Opial property. $\square$

References

[1] Altay B, Başar F. On some Euler sequence spaces of non-absolute type. Ukrainian Math J, 1999, **57**: 1–17

[2] Altay B, Başar F. Mursaleen M. On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I. Inform Sci, 2006, **10**: 1450–1462

[3] Altay B, Başar F. On the paranormed Riesz sequence spaces of non-absolute type. Southeast Asian Bull Math, 2002, **26**: 701–715

[4] Altay B, Başar F. Some paranormed Riesz sequence spaces of non-absolute type. Southeast Asian Bull Math, 2006, **30**: 591–608

[5] Başar F, Altay B. Matrix mappings on the space $bs(p)$ and its α-, β- and γ- duals. Aligarh Bull Math, 2002, **21**(1): 79–91

[6] Başar F. Infinite matrices and almost boundedness. Boll Un Mat Ital, 1992, **6**(7): 395–402

[7] Başar F, Altay B. On the space of sequences of p-bounded variation and related matrix mappings. Ukrainian Math J, 2003, **55**(1): 136–147

[8] Choudhary B, Mishra S K. On Köthe-Toeplitz duals of certain sequence spaces and their matrix transformations. Indian J Pure Appl Math, 1993, **24**(5): 291–301

[9] Cui Y, Hudzik H. On the uniform Opial property in some modular sequence spaces. Funct Approx Comment, 1998, **26**: 93–102

[10] Demiriz S, Çakan C. On some new paranormed Euler sequence spaces and Euler core. Acta Math Sin, English Series, 2010, **26**(7): 1207–1222

[11] Diestel J. Geometry of Banach Spaces. Selected Topics. Berlin: Springer-Verlag, 1975

[12] Grosse-Erdmann K G. Matrix transformations between the sequence spaves of Maddox. J Math Anal Appl, 1993, **180**: 223–238

[13] Gossez J P, Lami E D. Some geometric properties related to fixed point theory for nonexpensive mappings. Pacific J Math, 1972, **40**: 565–573

[14] Kara E E, Öztürk M, Başarır M. Some topological and geometric properties of generalized Euler sequence space. Mathematica Slovaca, 2010, **60**(3): 1–14

[15] Kızmaz H. On certain sequence spaces. Canad Math Bull, 1981, **24**(2): 169–176

[16] Khamsi M A. On Uniform Opial condition and Uniform Kadec-Klee property in Banach and metric spaces. Nonlinear Anal, 1996, **26**(10): 1733–1748

[17] Lascarides C G, Maddox I J. Matrix transformations between some classes of sequences. Proc Camb Phil Soc, 1970, **68**: 99–104

[18] Maddox I J. Elements of Functional Analysis. 2nd ed. Cambridge: Cambridge University Press, 1988

[19] Maddox I J. Paranormed sequence spaces generated by infinite matrices. Proc Camb Phios Soc. 1968, **64**: 335–340

[20] Maligranda L. Orlicz Spaces and Interpolation. Ponzan: Ins Math, Polish Academy of Sciences, 1989

[21] Malkowsky E. Recent results in the theory of matrix transformations in sequence spaces. Mat Vesnik, 1997, **49**(1): 187–196

[22] Mursaleen M, Başar F, Altay B. On the Euler sequence spaces which include the spaces ℓ_p an ℓ_∞ II. Nonlinear Anal, 2006, **65**: 707–717

[23] Nakano H. Modulared sequence spaces. Proc Jpn Acad, 1951, **27**(2): 508–512

[24] Opial Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull Amer Math Soc, 1967, **73**: 591–597

[25] Prus S. Banach spaces with uniform Opial property. Nonlinear Anal, 1992, **18**(8): 697–704

[26] Polat H, Başar F. Some Euler of difference sequences of order m. Acta Math Sci, 2007, **27B** (2): 254–266

[27] Simons S. The sequence spaces $\ell(p_v)$ and $m(p_v)$. Proc London Math Soc, 1965, **15**(3): 422–436