ORIGINAL PAPER

ANALYTICAL EVALUATION OF HOLTSMARK DISTRIBUTION OF ENERGIES AND ITS ROLE IN PLASMA MICROFIELDS

EBRU COPUROGLU[1], TURAL MEHMETOGLU[2]

Manuscript received: 10.02.2019; Accepted paper: 12.05.2019;
Published online: 30.06.2019.

Abstract. *The distribution of electric microfields influences many elementary processes in plasma as well as governs a number of its optic properties. It is well known that the Holtsmark distribution was first used for plasma microfields. Because the stars are consisting of plasmas, the Holtsmark distribution may give valuable information to the astrophysicists. With this spirit in mind, we have proposed an analytical expression for the calculation of Holtsmark distribution. The obtained results show that our approximation is in good agreement with literature data. The proposed method can be easily applied in plasma physical and astrophysical studies.*
Keywords: *Holtsmark distribution, plasma microfields, microfield calculations.*

1. INTRODUCTION

To the knowledge, plasma on microscales has strong electric fields because of the separation of two charges. Therefore, these electric fields called as microfields. The properties of microfields can be evaluated by the help of hydrogen like atoms placed in plasma. So, we can experience the Stark and Zeeman effects in electric and magnetic fields [1, 2]. Despite this and relevant approximations, the microfield calculations still sustain its complexity and kinetic problems. From this point of view, in literature both theoretical and semiemprical studies have been performed to obtain microfield distributions in plasmas [3-16]. As an example Rydberg lines in the ion spectra in dense plasma have been first investigated by Inglis and Teller in 1939 [6]. Also, by computing coupled Coulomb systems, the Baranger and Moser have given an useful approximation for microfield plasma including correlations between the particles [11]. The studies of dynamic and statistical properties of the electric microfield was also discussed in [1, 17]. It is clear that the problems of electric microfield plasma calculations are directly connected with the mathematical modeling.

Plasma microfield is explained by the electric field which is produced by the plasma ions and electrons near a radiating atom. The Holtsmark distribution plays a fundamental role in microfield plasma theory. It is clear that Holtsmark distribution has been used to investigate a plasma microfield which has been first performed by J. Holtsmark [3]. He studied electric microfield distributions by considering an one-component model which neglects particle correlations. The Holtsmark distribution became a physically significant solution of static microfield problem. The most important characteristic of this distribution is universality and depending only on isotropy of distribution function. And this function can be easily obtained by the Coulomb law of electric field.

[1] Gaziosmanpaşa University, Faculty of Arts and Sciences, Department of Physics, Tokat, Turkey.
E-mail: ebrucopuroglu@gmail.com
[2] Amasya University, Tasova Vocational School, Amasya, Turkey.

The Holtsmark distribution helps us to evaluate the probability density in ideal plasma for given location. So, an accurate definition of Holtsmark distribution can be a powerful tool to study stars consists of plasmas in astrophysics. A plasma has two types of microfields which are caused by ions and electrons. And so the both microfields have Holtsmark distributions [1].

In this study we have given an alternative, simple and full analytical method for the Holtsmark distribution which leads us to evaluate microfields of plasmas accurately. This study enables us to calculate the Holtsmark distribution using series expansion method in a simple and accurate way.

2. THEORY AND BASIC FORMULAS

The Holtsmark model uses the distribution of a sum of the elementary fields occurring from immobile ions microfields in plasma, following as [17]:

$$F=\sum_{i=1}^{N}F_i=-\sum_{i=1}^{N}\frac{e_i r_i}{r_i^3} \tag{1}$$

It can be seen that the probability is equal to the configuration space fraction. By the use of configuration space average we can separate this fraction as following:

$$W\left(F\right)=\left\langle \delta\left(F-e\sum_{k=1}^{N}\frac{r_k}{r_k{}^3}\right)\right\rangle=\frac{1}{\left(2\pi\right)^3}\int d\rho e^{i\rho F}\left\langle \exp\left(-ie\rho\sum_{k=1}^{N}\frac{r_k}{r_k{}^3}\right)\right\rangle \tag{2}$$

where function is the average function over configuration space. By considering distribution of mutually uncorrelated ions and some limitations we can get the following formula:

$$\begin{aligned}\left\langle \exp\left(-ie\rho\sum_{k=1}^{N}\frac{r_k}{r_k{}^3}\right)\right\rangle &=\int\ldots\int\exp\left(-ie\rho\sum_{k=1}^{N}\frac{r_k}{r_k{}^3}\right)\prod_{k=1}^{N}\frac{dr_k}{V}\\ &=\left(\int\exp\left(-ie\frac{\rho r}{r^3}\right)\frac{dr}{V}\right)^N\\ &=\exp(-N(\lambda e\rho)^{3/2})\end{aligned} \tag{3}$$

Here is the volume of the system and $\lambda=2\pi(4/15)^{2/3}$. By substituting (3) in (2) we get the following formula:

$$W\left(F\right)dF\equiv W\left(F\right)4\pi F^2 dF=H\left(\frac{F}{F_0}\right)\frac{dF}{F_0} \tag{4}$$

In Eq. (4), F_0 is the ion microfield and $H\left(x\right)$ is the Holtsmark function described as, respectively:

$$F_0 = \lambda e N^{2/3} \tag{5}$$

$$H(x) = \frac{2x}{\pi}\int_0^{\infty} \mathrm{y}\sin(xy)\exp(-y^{3/2})dy \tag{6}$$

where $x = F / F_0$, e and N are the charge and density of a particle, respectively.

Also, by using Mathematica integration procedure we have obtained following formula for Holtsmark distribution as:

$$\begin{aligned} W_H(x) = \frac{4x}{729\pi}\Bigg(& 28x^3\Gamma\left(-\frac{2}{3}\right){}_2F_3\left(\left[\frac{13}{12},\frac{19}{12}\right]:\left[\frac{2}{3},\frac{7}{6},\frac{3}{2}\right]:-\frac{4x^6}{729}\right) \\ & +22x^5\Gamma\left(\frac{2}{3}\right){}_2F_3\left(\left[\frac{17}{12},\frac{23}{12}\right]:\left[\frac{4}{3},\frac{3}{2},\frac{11}{6}\right]:-\frac{4x^6}{729}\right) \\ & +243x\,{}_3F_4\left(\left[\frac{3}{4},1,\frac{5}{4}\right]:\left[\frac{1}{3},\frac{2}{3},\frac{5}{6},\frac{7}{6}\right]:-\frac{4x^6}{729}\right)\Bigg) \end{aligned} \tag{7}$$

where ${}_qF_p$ are the well known special functions called as hypergeometric functions [18].

In this study, we have used the expansion of function appearing in Eq. (6) for the determination of Holtsmark distribution. It is clear that the expansion of can be written in following form [18]:

$$\sin(x) = \sum_{i=0}^{\infty}(-1)^i \frac{x^{2i+1}}{(2i+1)!} \tag{8}$$

Taking into account Eq. (8) in Eq. (6), we obtain following formula:

$$W_H(x) = (-1)^i \frac{2x^{2i+2}}{\pi(2i+1)!}\int_0^{\infty} y^{2i+2}\exp\left(-y^{3/2}\right)dy \tag{9}$$

The integral part of Eq. (9) can be easily expressed by Gamma functions. So, our formula reduces to the following form:

$$W_H(x) = \frac{2}{3}\lim_{N\to\infty}\sum_{i=0}^{N}(-1)^i \frac{2x^{2i+2}}{\pi(2i+1)!}\Gamma\left(\frac{4i+6}{3}\right) \tag{10}$$

here N is the upper limit of summation and $\Gamma(\alpha)$ is the common used Gamma functions defined as [19-21]:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt. \tag{11}$$

By using above mentioned procedure we have established a new, simple and analytical expression for Holtsmark distribution.

3. NUMERICAL RESULTS AND DISCUSSION

The main goal of this study is to examine the Holtsmark distribution of energies full analytically. From this point of view we have used the series expansion method for the trigonometric functions occurring in Holtsmark distribution function. It is well known that in literature the Holtsmark distribution is generally evaluated with numerical algorithms. So giving reliable analytical formulas for the calculation of Holtsmark distribution is very important. For this purpose, we have presented an analytical formula and performed a program by using Mathematica 7 programming language. The test results have been compared with numerical integration procedures and available literature data. As can be seen from Table 1, the analytical approximation obtained from this study is found closer to the Mathematica numerical data which ensures the accuracy of given formula. Notice that obtained formula provides high accuracy for values of $x < 6$. In these calculations the upper limit of summation N is taken up to 150. There is no restriction on increasing N and for yielding more accuracy we can use upper limits.

Table 1. Comparison results of Holtsmark distribution

x	Numerical results [13]	Results obtained from Eq.(7)	Results obtained from Eq.(10)
0	0	0	0
0.5	0.9459614695 E-1	0.9459614698 E-1	0.9459614698 E-1
1	0.2712208070	0.2712208070	0.2712208070
1.5	0.3635663742	0.3635663743	0.3635663743
2	0.3369387826	0.3369387827	0.3369387827
2.5	0.2553681350	0.2553681350	0.2553681350
3	0.1760629272	0.1760629272	0.1760629272
3.5	0.1183649017	0.1183649017	0.1183649017
4	0.8067354136 E-1	0.8067354136 E-1	0.8067354136 E-1
4.5	0.5667129991 E-1	0.5667129993E-1	0.5667132186 E-1
5	0.4118023717 E-1	0.4118023719 E-1	0.4118023719 E-1

The comparison of our results with Mathematica numerical results has been given in Fig. 1.

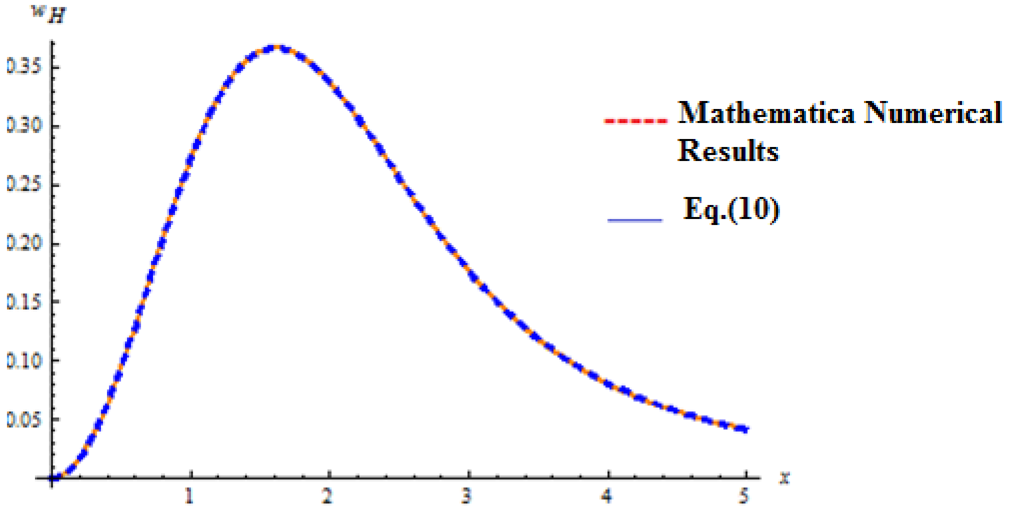

Figure 1. Comparison of Holtsmark distribution as a function of x with respect to Mathematica numerical results and Eq.(6)

To demonstrate the good behavior of our approximation with respect to the change of x we have plotted Holtsmark distribution as a function of x. It is understood from the graphic that our method is quite satisfactory. According to the computational results we can say that our approximation is simple, accurate and easily applied to the physical problems.

4. CONCLUSION

In conclusion, we have introduced a new full analytical formula of the Holtsmark distribution with the help of series expansion method. We noticed that the algorithm in this paper is of completely general and can be easily applied to plasma physic studies. The chosen method and its applications are comparable with literature data.

REFERENCES

[1] Romanovsky, M. Y., Ebeling, W., *Contrib. Plasma Phys*. **46**, 195, 2006.
[2] Ecker, G., *Theory of Fully Ionized Plasmas*, Academic Press, NY, 1972.
[3] Holtsmark, J., *Ann. Physik*, **58**, 577, 1919.
[4] Tonks, L., Langmuir, I., *Phys. Rev*. **55**, 195, 1929.
[5] Lanczos, E., *Zeitschrift für Physik*, **62**, 518, 1930.
[6] Inglis, D. R., Teller, E., *Astrophys J.*, **90**, 439, 1939.
[7] Feller, W., Wiley, J and Sons, *An introduction to the probability theory and its applications*, NY, 1970.
[8] Griem, H. R., *Spectral line broadening by plasmas*, Academic Press, N-Y, 1976.
[9] Unsold, A., *Physik der Sternatmosphären*, Springer, Berlin, 1955.
[10] Klimontovich, Yu. L., *The kinetic theory of electromagnetic processes*, Springer, Berlin, Heidelberg, NY, 1983.
[11] Baranger, M., Mozer, B., *Phys. Rev*. **115**, 521, 1959; **118**, 626, 1960.

[12] Avni, R., Miralai, S.F., Prevot, F., Morvan, D., Amoroux, J., Nickel, H., *Plasma Chem. Plasma Process*, **17**, 4, 1997.
[13] Dubinova, I. D., Dubinov, A. E., *Contrib. Plasma Phys.*, **47**, 445, 2007.
[14] Mamedov, B. A., *Astroparticle Physics,* **64**, 2015.
[15] Mamedov, B. A., *Physics of Plasmas*, **18**, 2011.
[16] Mamedov, B. A., *Contribution to Plasma Physics*, **49**, 36, 2009.
[17] Shafranov, V. D., *Reviews of Plasma Physics*, Vol. 22, Springer Science & Business Media, 2012.
[18] http://mathworld.wolfram.com/GeneralizedHypergeometricFunction.html.
[19] Gradshteyn, I. S., Ryzhik, I. M., *Tables of Integrals, Sums, Series and Products*, 4th ed., Academic Press, New York, 1980.
[20] Guseinov, I. I., Mamedov, B. A., *JQSRT*, **2**, 251, 2006.
[21] Çopuroğlu, E., Mehmetoğlu, T., *IEEE Trans. Electron Devices*, **62**, 1580, 2015.