

Note

Combinatory weak reduction in lambda calculus

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Abstract

Combinatory logic claims to do the same work as λ -calculus but with a simpler language and a simpler reduction process. In a sense this claim is true: the classical reduction process in λ -calculus is indeed more complex than that in combinatory logic. But by changing its definition only slightly one can define in λ -calculus a perfect analogue of combinatory reduction. This analogue was first formulated 30 years ago but it is still not as well known as it deserves, so in the present purely expository paper we shall try to make it more accessible. We shall discuss its definition, motivation and its neat relation to substitution. © 1998—Elsevier Science B.V. All rights reserved

1. Introduction

In λ -calculus the concept of abstraction $\lambda x.M$ is one of the basic term-building operations, but in combinatory logic (CL) one constructs abstraction terms using atomic combinators, often S, K and I. This difference leads to different definitions of reduction in each system. In λ -calculus the classic reduction is β -reduction \triangleright_β , defined by replacing terms $(\lambda x.M)N$ called redexes. In CL the usual reduction is weak reduction \triangleright_w , defined by replacing certain redexes associated with the combinators. (Precise definitions will be given below.)

Both these reductions have many properties in common, for example, the Church–Rosser and Standardisation theorems. But the proofs of these properties are always easier in CL. One reason is the rather technical fact that if a reduction $X \triangleright_w Y$ is made in CL and X contains a set R of non-overlapping redexes, then in Y the residuals of the members of R (defined in a certain sense) will not overlap each other. This is not

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true for λ -terms, and for this reason any proof that depends on analysing the effect of a reduction on a set of redexes will be harder in λ -calculus.¹

Because of this it is natural to wish to modify the reduction \triangleright_β to make it correspond more closely to \triangleright_w and to simplify its properties. It is not very widely known that this can actually be done. A suitable definition of a weak λ -reduction was given as long ago as 1968 by William Howard in [7, p. 446], and was repeated and discussed in [5, Section 2]. But in [7] it was buried in an account of other topics and in [5] its basic properties were merely stated without proofs. Our aim here is to give a clearer account of this modified reduction than either of those sources.²

Since its first application in [7], weak λ -reduction has been applied in [8], see especially Section 2 there, and in [1]. We hope that the present account will stimulate further interest.

2. Basic definitions

We shall assume the reader has met λ -calculus and combinatory logic before, and shall just recall some main points below; a fuller introduction can be found in [2, Chs. 2 and 7], or [4, Chs. 2 and 4], or [6, Chs. 1 and 2]. The notation of [6] will be used here. In particular, the result of substituting N for all free occurrences of x in M , and changing bound variables if necessary to avoid clashes, will be called $[N/x]M$.

Definition 2.1. In λ -calculus, a β -redex is any term $(\lambda x.M)N$, its *contractum* is $[N/x]M$, and a β -contraction is the process of replacing a β -redex by its contractum. If a λ -term P changes to Q by a finite (perhaps empty) series of β -contractions, and perhaps some changes of bound variables, one says

$$P \triangleright_\beta Q.$$

The following alternative definition of \triangleright_β will also play a role.

Definition 2.2. In λ -calculus, define a relation $P \triangleright_\beta Q$ inductively by the following clauses:

- (ρ) $P \triangleright_\beta P$,
- (α) $\lambda x.M \triangleright_\beta \lambda y.[y/x]M$ if $y \notin FV(M)$
- (β) $(\lambda x.M)N \triangleright_\beta [N/x]M$,
- (τ) $P \triangleright_\beta Q, Q \triangleright_\beta T \Rightarrow P \triangleright_\beta T$,

¹ For example, although a typical modern proof of the Church–Rosser theorem for λ -calculus is not all that complicated (for example, see [2, Section 3.2] or [4, Section 3.3]), that of the same theorem for CL is even simpler (see, for example, the original proof in [9, Part 1, pp. 140–146]). Neither proof mentions residuals explicitly, but their difference can be viewed as being due essentially to the above property.

² The present expository paper is a condensation of part of the dissertation [3]. For financial support which made this work possible, the first author is very grateful to Gaziosmanpaşa Üniversitesi, Tokat, Turkey.

- (μ) $P \triangleright_{\beta} Q \Rightarrow MP \triangleright_{\beta} MQ$,
- (ν) $P \triangleright_{\beta} Q \Rightarrow PM \triangleright_{\beta} QM$,
- (ξ) $P \triangleright_{\beta} Q \Rightarrow \lambda x.P \triangleright_{\beta} \lambda x.Q$.

Lemma 2.3. *Definitions 2.1 and 2.2 define the same relation $P \triangleright_{\beta} Q$.*

Now, turn to combinatory logic (CL). We shall assume that the basic combinators are **S**, **K** and **I**. The commonly used reducibility relation is called *weak reducibility* or \triangleright_w , and it may be defined in either of the following two equivalent ways:

Definition 2.4. In CL a (*weak*) *redex* is any CL-term **S***XYZ* (whose contractum is *XZ*(*YZ*)), or **K***XY* (whose contractum is *X*), or **I***X* (whose contractum is *X*), and the process of replacing a redex by its contractum is called a *contraction*. If a CL-term *X* changes to *Y* by a finite (perhaps empty) series of contractions, one says

$$X \triangleright_w Y.$$

Definition 2.5. In CL, define a relation $X \triangleright_w Y$ inductively by the following clauses:

- (ρ) $X \triangleright_w X$,
- (**S**,**K**,**I**) **S***XYZ* \triangleright_w *XZ*(*YZ*), **K***XY* \triangleright_w *X*, **I***X* \triangleright_w *X*,
- (τ) $X \triangleright_w Y, Y \triangleright_w Z \Rightarrow X \triangleright_w Z$,
- (μ) $X \triangleright_w Y \Rightarrow UX \triangleright_w UY$,
- (ν) $X \triangleright_w Y \Rightarrow XU \triangleright_w YU$.

Lemma 2.6. *Definitions 2.4 and 2.5 define the same relation $X \triangleright_{\beta} Y$.*

Between CL and λ -calculus there is a correspondence determined by the following two mappings (see [6, Ch. 9 for details]).

Definition 2.7. To every CL-term *X*, a λ -term called X_{λ} is assigned as follows:

- $x_{\lambda} \equiv x$,
- $(XY)_{\lambda} \equiv (X_{\lambda} Y_{\lambda})$,
- $I_{\lambda} \equiv \lambda x.x, \quad K_{\lambda} \equiv \lambda x y.x, \quad S_{\lambda} \equiv \lambda x yz.xz(yz)$.

Definition 2.8. To every λ -term *M*, a CL-term M_H is assigned as follows:

- $x_H \equiv x$,
- $(MN)_H \equiv (M_H N_H)$,
- $(\lambda x.M)_H \equiv \lambda^*x.(M_H)$,

where $\lambda^*x.X$ is defined for all CL-terms *X* by the following algorithm ([6, Definition 2.14]):

- (a) $\lambda^*x.X \equiv KX$ if $x \notin FV(X)$,

- (b) $\lambda^*x.x \equiv 1$,
 (c) $\lambda^*x.Ux \equiv U$ if $x \notin FV(U)$,
 (f) $\lambda^*x.UV \equiv S(\lambda^*x.U)(\lambda^*x.V)$ if none of (a)–(c) applies.

The following well-known facts will be used later.

Lemma 2.9. For all CL-terms X, Y, Z and all λ -terms M, N :

- (i) $(\lambda^*x.X)Y \triangleright_w [Y/x]X$;
 (ii) $X \triangleright_w Y \Rightarrow [X/v]Z \triangleright_w [Y/v]Z$;
 (iii) $(X_\lambda)_H \equiv X$.
 (iv) $([N/x]M)_H \equiv [N_H/x](M_H)$.

Proof. (i) and (ii) are easy, (iii) is [6, Lemma 9.8], and (iv) is [6, Lemma 9.10]. \square

3. Weak λ -reduction

We shall now define a λ -reduction analogous to combinatory weak reduction \triangleright_w , by modifying the first definition of \triangleright_β . To see the main difference between \triangleright_β and \triangleright_w , consider for example $P \equiv \lambda y.(\lambda x.xy)z$, $R \equiv (\lambda x.xy)z$ and $Q \equiv \lambda y.zy$. Then

$$P \equiv \lambda y.((\lambda x.xy)z) \triangleright_\beta \lambda y.zy \equiv Q. \quad (1)$$

To translate this reduction into CL it is natural to use the H -mapping defined previously; but this fails, as it is not true that $P_H \triangleright_w Q_H$. In fact, by Definition 2.8,

$$\begin{aligned} R_H &\equiv (\lambda^*x.xy)z \equiv S(Ky)z, \\ P_H &\equiv \lambda^*y.R_H \equiv S(S(K(SI))K)(Kz), \\ Q_H &\equiv \lambda^*y.zy \equiv z, \end{aligned}$$

and P_H does not weakly reduce to Q_H . (It does not weakly reduce at all, because it contains no weak redex.)

The problem is that P contains a λy which binds the free y in R . If this did not happen, for example, if $P \equiv \lambda w.(\lambda x.xy)z$ for some new variable w , then

$$P \equiv \lambda w.((\lambda x.xy)z) \triangleright_\beta \lambda w.zy, \quad (2)$$

and P_H would weakly reduce to $(\lambda w.zy)_H$ as follows:

$$P_H \equiv KR_H \equiv K(SI(Ky)z) \triangleright_w K(zy) \equiv \lambda^*w.zy.$$

This suggests that a β -reduction could be called “weak” when the variables in its redexes are not bound by λ ’s outside them. The following definition formalises this idea. It is due to William Howard ([7, p. 446, “restricted reductions”]).

Definition 3.1 (*Weak λ -reduction*). A particular occurrence of a β -redex R in a λ -term P will be called *weak in P* iff no variable-occurrence free in R is bound in P . A *weak β -contraction in P* is the contraction of a β -redex-occurrence that is weak in P . Iff a λ -term P is changed to Q by a finite (perhaps empty) series of weak β -contractions, and perhaps some changes of bound variables, we shall say

$$P \triangleright_{w\lambda} Q.$$

Note that the property of being weak depends on P as well as on R . For example, the contraction (2) above is weak but (1) is not, although the same redex is contracted in both cases.

Note also that $\triangleright_{w\lambda}$ does not satisfy the “rule”

$$(\xi) \quad M \triangleright N \Rightarrow \lambda x.M \triangleright \lambda x.N.$$

But this failure should be welcomed if we are trying to imitate CL, because weak reduction in CL does not satisfy the corresponding “rule”

$$(\xi_{CL}) \quad X \triangleright Y \Rightarrow \lambda^*x.X \triangleright \lambda^*x.Y.$$

The precise correspondence between $\triangleright_{w\lambda}$ in λ -calculus and \triangleright_w in CL is as follows:

Proposition 3.2. *For all CL-terms X and Y , and all λ -terms P and Q ,*

- (i) $X \triangleright_w Y \Leftrightarrow X_\lambda \triangleright_{w\lambda} Y_\lambda$,
- (ii) $P \triangleright_{w\lambda} Q \Rightarrow P_H \triangleright_w Q_H$.

Proof. For (i) “ \Rightarrow ”, use induction on Definition 2.5. The only non-trivial clause is (S,K,I), and this translates into λ -calculus as

$$\begin{aligned} & (\lambda x y z. xz(yz))X_\lambda Y_\lambda Z_\lambda \triangleright_\beta X_\lambda Z_\lambda (Y_\lambda Z_\lambda), \\ & (\lambda x y. x)X_\lambda Y_\lambda \triangleright_\beta X_\lambda, \\ & (\lambda x. x)X_\lambda \triangleright_\beta X_\lambda \end{aligned}$$

and all six contractions involved in these reductions are easily seen to be weak.

For (i) “ \Leftarrow ”, use (ii) and Lemma 2.9(iii).

For (ii), see Corollary 4.8. \square

Thus, $\triangleright_{w\lambda}$ corresponds closely to \triangleright_w in CL. The correspondence is not perfect, because the converse of Proposition 3.2(ii) is not true: $P_H \triangleright_w Q_H$ does not imply $P \triangleright_{w\lambda} Q$. (A counterexample is $P \equiv \lambda y. xy$, $Q \equiv x$, with $P_H \equiv x \equiv Q_H$.) But as discussed in [5, Section 2], it is probably as close as one can get without losing desirable technical properties such as the Church–Rosser theorem.

We shall now state that theorem for $\triangleright_{w\lambda}$, but shall not burden the reader with details of its proof.

Theorem 3.3 (Church–Rosser Theorem). *If $P \triangleright_{w\lambda} M$ and $P \triangleright_{w\lambda} N$, then there exists a λ -term T such that*

$$M \triangleright_{w\lambda} T, \quad N \triangleright_{w\lambda} T.$$

Proof-outline. In the proof for β -reduction in [6, Appendix 1, Theorem A.1.2], two new reductions $M \triangleright_{\beta} T$ and $N \triangleright_{\beta} T$ are constructed by using what are called *residuals* of the redexes involved in the given reductions $P \triangleright_{\beta} M$ and $P \triangleright_{\beta} N$. To make this proof apply to $\triangleright_{w\lambda}$, it is enough simply to prove that the residuals of a weak redex are weak. The next lemma states the latter fact in more detail.

(The Church–Rosser proofs for \triangleright_{β} in [2, Section 3.2] and [4, Section 3.3] can also be easily adapted to $\triangleright_{w\lambda}$.) \square

Lemma 3.4 (Hindley [5, Section 2, p. 172]). *If two β -redex-occurrences R and S are weak in a λ -term P , and contracting R changes P to P' , then the residuals of S in P' are weak in P' , where residuals are defined as in [6, Appendix 1, Definition A.1.3].*

Proof. Check the cases in the definition of residuals. \square

Note 3.5. By the way, an alternative motivation of the definition of $\triangleright_{w\lambda}$ is given by considering a λ -term $P \equiv (\lambda x.R_1)R_2$, where R_1 and R_2 are redexes and R_1 contains one free occurrence of x . Let Q be the result of contracting P :

$$P \equiv (\lambda x.R_1)R_2 \triangleright_{\beta} [R_2/x]R_1 \equiv Q.$$

Then this contraction has changed the non-overlapping pair R_1, R_2 of redexes in P into an overlapping pair in Q . As remarked in the Introduction, this cannot happen in CL (try it and see!), so this is a situation that a λ -analogue of combinatory reduction must avoid. Its cause is the free x in R_1 being bound in P , and this is just what Definition 3.1 forbids.

Note 3.6. If one is only interested in reducing closed terms, Definition 3.1 can be simplified to say that a β -redex is weak iff it contains no free variables. In this form the definition has appeared in [8, p. 182].

4. Weak reduction and substitution

As suggested in [5, p. 172], the difference between weak and ordinary λ -reduction can be expressed very neatly in terms of the difference between substitution and replacement.

For *substitution*, $[N/x]M$ is the result of substituting N for all free occurrences of x in M and changing bound variables if necessary to avoid clashes. Its detailed definition is in [6, Definition 1.11] or [4, Section 2.3, p. 15].

The operation we shall call *replacement* is the same as substitution but without changing bound variables. It is usually defined as follows (see [2, p. 29 Definition 2.1.18], or [4, pp. 10–11]).

Definition 4.1. First, add to the language of λ -calculus a new atom called $[]$; terms containing this atom are called *contexts*, and each occurrence of $[]$ in a context is

called a *hole*. The notation for an arbitrary context is $C[\]$; if every hole in $C[\]$ is replaced by a term N the result is called

$$C[N].$$

For example, let $C[\] \equiv \lambda y.y[\]y$ and $N \equiv xy$, then

$$C[N] \equiv \lambda y.y(xy)(xy)y.$$

Compare this with substitution: in $C[N]$ the λy has been allowed to “capture” the free y in N and hold it bound, but in a substitution one changes λy to λz to prevent this happening:

$$[N/x](\lambda y.yxy) \equiv \lambda z.z(xy)(xy)z.$$

However, in the special case when N has no free variables that would become bound in $C[N]$, the clause in the definition of substitution about changing bound variables is never used, and substitution and replacement are the same; the following lemma expresses this fact formally.

Lemma 4.2. *If $C[\]$ is any context in λ -calculus and no variable-occurrence free in N is bound in $C[N]$, and x does not occur in $C[\]$, then*

$$[N/x](C[x]) \equiv C[N].$$

Now, β -reduction can be defined in terms of replacement: in fact it is easy to see that the three rules (μ) , (ν) and (ζ) in Definition 2.2 are equivalent to the replacement rule in the next definition.

Definition 4.3. Define a relation $P \triangleright_{\beta} Q$ in λ -calculus inductively by the following clauses: (ρ) , (α) , (β) , (τ) as in Definition 2.2, plus

$$(\text{Rep}) \quad P \triangleright_{\beta} Q \Rightarrow C[P] \triangleright_{\beta} C[Q] \quad \text{for all contexts } C[\].$$

Lemma 4.4. *Definition 4.3 defines the same relation \triangleright_{β} as Definitions 2.1 and 2.2.*

On the other hand, weak λ -reduction can be defined in terms of substitution: we shall define a relation $>_{w\lambda}$ below using a substitution rule and then show it is identical to $\triangleright_{w\lambda}$.

Definition 4.5. Define a relation $P >_{w\lambda} Q$ in λ -calculus inductively by clauses (ρ) , (α) , (β) , (τ) as in Definition 2.2, plus

$$(\text{Sub}) \quad P >_{w\lambda} Q \Rightarrow [P/x]T >_{w\lambda} [Q/x]T \quad \text{for all terms } T.$$

Proposition 4.6. *Definitions 4.5 and 3.1 define the same relation; i.e.*

$$P >_{w\lambda} Q \Leftrightarrow P \triangleright_{w\lambda} Q.$$

Proof. For “ \Rightarrow ”, we must show that if $P >_{w\lambda} Q$ then P changes to Q by a series of weak contractions. For clauses (ρ) , (α) , (β) and (τ) in Definition 4.5 this is trivial. For (Sub): suppose P changes to Q by a series of weak contractions, we must show that $[P/x]T$ changes to $[Q/x]T$ by such a series. But this will follow from Lemma 4.7 below.

For “ \Leftarrow ”, it is enough to show that if P changes to Q by one weak contraction, then $P >_{w\lambda} Q$. Let $P \equiv C[R]$ where R is weak in P , and let $Q \equiv C[R^*]$ where R^* is the contractum of R . Then, by Lemma 4.2,

$$P \equiv [R/x](C[x]),$$

where x does not occur in $C[\]$. And $FV(R^*) \subseteq FV(R)$, so by the same lemma,

$$Q \equiv [R^*/x](C[x]).$$

But $R >_{w\lambda} R^*$ by (β) in Definition 4.5, therefore $P >_{w\lambda} Q$ by (Sub). \square

Lemma 4.7. *If a β -redex-occurrence R is weak in a λ -term P , then the corresponding occurrences of R in $[P/x]T$ are weak, for every λ -term T .*

Proof (Induction on T). The only non-trivial case is when $T \equiv \lambda v.M$ for some v and M . In this case let u be a variable-occurrence free in R . Then u is free in P , because R is weak in P . Hence, by the definition of substitution (for details see [6, Definition 1.11], or [4, Section 2.3, p. 15]), if there is a λu in M whose scope contains a free occurrence of v , that λu would be changed to avoid binding the free u in P . Thus, u must be free in $[P/x](\lambda v.M)$. \square

Corollary 4.8. *Part (ii) of Proposition 3.2 holds; that is, for all λ -terms P, Q ,*

$$P \triangleright_{w\lambda} Q \Rightarrow P_H \triangleright_w Q_H.$$

Proof. By Proposition 4.6 (which is independent of 3.2(ii)) we can use induction on the clauses of Definition 4.5. The cases (ρ) , (α) and (τ) are trivial.

For (β) , suppose $P \equiv (\lambda x.M)N \triangleright_{w\lambda} [N/x]M \equiv Q$. Then

$$\begin{aligned} P_H &\equiv (\lambda^* x.M_H)N_H \triangleright_w [N_H/x]M_H \quad \text{by Lemma 2.9(i),} \\ &\equiv Q_H \quad \text{by Lemma 2.9(iv).} \end{aligned}$$

For (Sub), assume that $P \triangleright_{w\lambda} Q$ and $P_H \triangleright_w Q_H$. Then, for any T and x ,

$$\begin{aligned} ([P/x]T)_H &\equiv [P_H/x]T_H \quad \text{by Lemma 2.9(iv),} \\ &\triangleright_w [Q_H/x]T_H \quad \text{by assumption and Lemma 2.9(ii),} \\ &\equiv ([Q/x]T)_H \quad \text{by Lemma 2.9(iv).} \quad \square \end{aligned}$$

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