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Oscillatory Behavior of Semi-canonical Nonlinear Neutral Differential Equations of Third-Order Via Comparison Principles

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Abstract
We are dealing here with the oscillatory behaviors of all solutions to a class of semicanonical third-order nonlinear functional differential equations involving both retarded and advanced type deviating arguments in the neutral term. Our main tool is to show that, under certain conditions, a semicanonical equation can be written as an equivalent equation in canonical form. The results presented in this paper are novel even when the studied equation is linear. We also provide several examples to demonstrate the applicability and versatility of our results.

Keywords Semi-canonical · Neutral · Comparison · Third-order · Oscillation

Mathematics Subject Classification 34C10 · 34K11 · 34K40

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1 Introduction

Differential equations (DEs) of third-order arise as mathematical models in a considerable amount of problems in mathematical biology, physics and engineering, for instance, in the study of equilibrium equations in the theory of elasticity, in research to obtain the traveling wave solutions of partial DEs that occur when describing the model formulation for reaction diffusion systems, in problems of the regulation of a steam turbine, in the investigation of the entry-flow phenomenon, which is a very important hydrodynamic problem in engineering [13, 20]. We refer to the above mentioned monographs of Padhi and Pati [20], and Greguš [13] for those who want to explore more about the applications of such equations.

In this work, we focus on the oscillation of all solutions of a class of the third order semi-canonical mixed neutral DEs of the form

$$\mathcal{L}z(t) + h(t)y^{\beta}\big(\zeta(t)\big) = 0, \quad t \geq t_0 > 0, \tag{E}$$

where the differential operator $\mathcal{L}$ is denoted by

$$\mathcal{L}z(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^{\gamma}\right)\right),$$

β and γ are the ratios of positive odd integers, and the neutral term $z(t)$ is defined to include both delayed and advanced arguments such that

$$z\,(t) := y\,(t) + r_1(t)y\big(\theta_1(t)\big) + r_2(t)y\big(\theta_2(t)\big). \tag{1.1}$$

Throughout the entire article, without further mention, the following assumptions are always supposed to hold:

(i) $h, r_1, r_2 \in C\,([t_0, \infty), \mathbb{R})$ with $r_1(t) \geq 1$, $r_2(t) > 0$, $r_1(t) \not\equiv 1$ for t large enough, and $h(t) \geq 0$ is not identically zero for large t;

(ii) $\theta_1, \theta_2, \zeta \in C\,([t_0, \infty), \mathbb{R})$, $\theta_1(t) \leq t \leq \theta_2(t)$, ζ is nondecreasing, θ_1 and θ_2 are strictly increasing, and $\lim_{t\to\infty} \theta_1\,(t) = \lim_{t\to\infty} \zeta\,(t) = \infty$;

(iii) the operator $\mathcal{L}$ is in semi-canonical form, namely,

$$\int_{t_0}^{\infty} \frac{1}{f(\eta)}\mathrm{d}\eta < \infty \text{ and } \int_{t_0}^{\infty} \left(\frac{1}{g(\eta)}\right)^{1/\gamma} \mathrm{d}\eta = \infty, \tag{1.2}$$

where $f, g \in C\left([t_0, \infty), \mathbb{R}^+\right)$.

Recall that a solution of Eq. (E) is a nontrivial real-valued function y satisfying Eq. (E) for $t \geq t_*$ for some $t_* \geq t_0$ such that $z \in C^1([t_*, \infty), \mathbb{R})$, $g(z')^{\gamma} \in C^1([t_*, \infty), \mathbb{R})$, and $f(g(z')^{\gamma})' \in C^1([t_*, \infty), \mathbb{R})$. We suppose that Eq. (E) admits such solutions, and the solutions that vanish identically in some neighborhood of $+\infty$ will not be in the evaluation of this manuscript. Such a nontrivial solution $y(t)$ of Eq. (E) is said to be oscillatory if $y(t)$ is neither eventually negative nor eventually positive, and otherwise,

$y(t)$ is termed nonoscillatory. If every solution is oscillatory, then the equation itself is called oscillatory.

Equation (E) is said to be in canonical form if

$$\int_{t_0}^{\infty} \frac{1}{f(\eta)} \mathrm{d}\eta = \infty \text{ and } \int_{t_0}^{\infty} \left(\frac{1}{g(\eta)}\right)^{1/\gamma} \mathrm{d}\eta = \infty, \tag{1.3}$$

and it is said to be in noncanonical form if

$$\int_{t_0}^{\infty} \frac{1}{f(\eta)} \mathrm{d}\eta < \infty \text{ and } \int_{t_0}^{\infty} \left(\frac{1}{g(\eta)}\right)^{1/\gamma} \mathrm{d}\eta < \infty. \tag{1.4}$$

If either (1.2) holds or

$$\int_{t_0}^{\infty} \frac{1}{f(\eta)} \mathrm{d}\eta = \infty \text{ and } \int_{t_0}^{\infty} \left(\frac{1}{g(\eta)}\right)^{1/\gamma} \mathrm{d}\eta < \infty, \tag{1.5}$$

then Eq. (E) is said to be in semi-canonical form [23].

The oscillation and asymptotic behavior of solutions for different classes of third-order and/or higher order functional differential, difference and dynamic equations is an important scientific research topic that has been studied extensively by many authors in last years; see, e.g., the papers [1–12, 17–19, 23–27, 29–31] for recent results on the literature.

In particular, the examination of canonical, noncanonical and semi-canonical forms of third-order neutral DEs and neutral dynamic equations on time scales has both theoretical and practical importance, because of their direct applications in science. For third-order neutral equations of canonical type, in [4, 8, 26], the authors discussed the oscillation of all solutions or oscillatory and asymptotic behavior of solutions subject to bounded neutral coefficients. Also, in [6, 7, 11, 29, 30], several comparison theorems and/or integral oscillation crieria have been obtained that can apply to unbounded neutral coefficients.

For noncanonical neutral equations of third-order, the literature regarding the problem of obtaining oscillation criteria is relatively scarce, we refer to the articles [5, 10], and cited studies therein. It should be pointed out that all these studies mentioned here are devoted to equations including bounded neutral coefficients.

There is very little research dealing with third-order semi-canonical functional DEs, because the work on these types of equations has only recently begun. Some important results on this subject can be found in [23, 24] for the delay and/or advanced DEs, in [8, 25] for neutral DEs, and in [12, 31] for dynamic equations on time scales. However, from the review of the literature, it is apparent that there seem to be no results regarding the oscillation of semi-canonical DEs of third-order with mixed neutral terms.

Therefore, the main goal of this study is to give an affirmative answer to the question of whether oscillation criteria can be obtained for these types DEs with mixed neutral terms. It is shown with three examples in the last section that the obtained results can be applied both to DEs containing bounded and/or unbounded neutral coefficients.

Since we are dealing with the third-order nonlinear semi-canonical neutral DEs of mixed type which has not been performed before, the results of this paper complement and extend the previous ones in [8, 23–25, 31].

2 Some Preliminary Lemmas

To facilitate writability, the following symbols will be used:

$$\mathcal{F}(t) := \int_t^\infty \frac{1}{f(\eta)}\mathrm{d}\eta, \quad \psi(t) := f(t)\mathcal{F}^2(t), \quad \varphi(t) := \frac{g(t)}{\mathcal{F}(t)},$$
$$\mathcal{H}(t) := h(t)\mathcal{F}(t), \quad \Psi(t) := \int_{t_0}^t \frac{1}{\psi(\eta)}\mathrm{d}\eta, \quad \mathcal{G}(t) := \int_{t_0}^t \left(\Psi(\eta)\varphi^{-1}(\eta)\right)^{1/\gamma}\mathrm{d}\eta,$$
$$\xi(t) := \theta_1^{-1} \circ \zeta(t), \quad \mu(t) := \theta_1^{-1} \circ \phi(t), \quad \phi \in C^1([t_0, \infty), \mathbb{R}),$$
$$\Omega(t) := \exp\left(\int_{t_1}^t \left(\frac{\Psi(\eta)}{\varphi(\eta)}\right)^{1/\gamma} \frac{1}{\mathcal{G}(\eta)}\mathrm{d}\eta\right) \text{ for } t \geq t_1 \text{ for some } t_1 \geq t_0, \text{ and}$$
$$\mathcal{R}(t) := \int_{\xi(t)}^{\mu(t)} \left(\frac{1}{\varphi(u)}\int_u^{\mu(t)} \frac{1}{\psi(v)}\mathrm{d}v\right)^{1/\gamma}\mathrm{d}u.$$

Moreover, it will also be assumed that the following assumptions are always satisfied for large t enough:

$$\mathcal{Q}_*(t) := \frac{1}{r_1(t)}\left[1 - \frac{1}{r_1(\theta_1^{-1}(t))}\frac{\Omega(\theta_1^{-1}(t))}{\Omega(t)} - \frac{r_2(t)}{r_1(\theta_1^{-1} \circ \theta_2(t))}\frac{\Omega(\theta_1^{-1} \circ \theta_2(t))}{\Omega(t)}\right] \geq 0$$

and

$$\mathcal{Q}_{**}(t) := \left[1 - \frac{1}{r_1(\theta_1^{-1}(t))} - \frac{r_2(t)}{r_1(\theta_1^{-1} \circ \theta_2(t))}\right]\frac{1}{r_1(t)} \geq 0,$$

where θ_1^{-1} shows the inverse of θ_1.

Theorem 2.1 *Assume that*

$$\int_{t_0}^\infty \frac{1}{\varphi^{1/\gamma}(t)}\mathrm{d}t = \infty. \tag{2.1}$$

Then the semi-canonical operator $\mathcal{L}$ can be written in canonical form as

$$\mathcal{L}z(t) = \frac{1}{\mathcal{F}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\mathcal{F}^2(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g(t)}{\mathcal{F}(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^\gamma\right)\right), \tag{2.2}$$

and this representation is unique.

Proof Direct calculation shows that

$$f(t)\mathcal{F}^2(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g(t)}{\mathcal{F}(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^{\gamma}\right)=f(t)\mathcal{F}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^{\gamma}\right)+g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^{\gamma}.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\mathcal{F}^2(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g(t)}{\mathcal{F}(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^{\gamma}\right)\right)=\mathcal{F}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^{\gamma}\right)\right),$$

which implies

$$\frac{1}{\mathcal{F}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\mathcal{F}^2(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g(t)}{\mathcal{F}(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^{\gamma}\right)\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^{\gamma}\right)\right).$$

Taking (2.1) into account, we get

$$\int_{t_0}^{\infty}\left(\frac{\mathcal{F}(t)}{g(t)}\right)^{1/\gamma}\mathrm{d}t=\infty,$$

and due to

$$\int_{t_0}^{\infty}\frac{1}{f(t)\mathcal{F}^2(t)}\mathrm{d}t=\int_{t_0}^{\infty}\mathrm{d}\left(\frac{1}{\mathcal{F}(t)}\right)=\lim_{t\to\infty}\left(\frac{1}{\mathcal{F}(t)}-\frac{1}{\mathcal{F}(t_0)}\right)=\infty,$$

we deduce that (2.2) is in canonical form. Since this canonical representation is unique (proved by Trench in [28]), the proof is complete. □

From Theorem 2.1, it follows that under condition (2.1), Eq. (E) can be written in the canonical form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)^{\gamma}\right)\right)+\mathcal{H}(t)y^{\beta}\big(\zeta(t)\big)=0,\quad t\geq t_0>0,\qquad\text{(EQ)}$$

and, immediately, the following results are obtained.

Theorem 2.2 *Under condition (2.1), $y(t)$ is a solution of semi-canonical neutral DE (E) iff it is a solution of the canonical neutral DE (EQ).*

Corollary 2.3 *Under condition (2.1), the semi-canonical neutral DE (E) has an eventually positive solution (EPS) iff canonical neutral DE (EQ) has an EPS.*

The above corollary significantly simplifies the investigation of solutions of Eq. (E) to be for (EQ). We will analyze only two classes of EPSs according to the following lemma (see, [16, Lemma 2]):

Lemma 2.1 *Assume that $y(t)$ is an EPS of (EQ) and let (2.1) be satisfied. Then, a $t_1 \in [t_0, \infty)$ can be found such that the corresponding $z(t)$ satisfies one of the following two cases for $t \geq t_1$:*

$$z(t) > 0, \quad \left(\varphi(z')^{\gamma}\right)(t) < 0, \quad \left(\varphi(z')^{\gamma}\right)'(t) > 0, \quad \left(\psi\left(\varphi(z')^{\gamma}\right)'\right)'(t) \leq 0,$$

and we specify that $z \in \mathcal{N}_0$ in this case,

$$z(t) > 0, \quad \left(\varphi(z')^{\gamma}\right)(t) > 0, \quad \left(\varphi(z')^{\gamma}\right)'(t) > 0, \quad \left(\psi\left(\varphi(z')^{\gamma}\right)'\right)'(t) \leq 0,$$

and we specify that $z \in \mathcal{N}_2$ in this case.

The proof of the next lemma proceeds similarly to the proofs in [25, Lemma 2.4] and [7, Lemma 2.3], therefore the details are not repeated here.

Lemma 2.2 *Assume that $y(t)$ is an EPS of (EQ) such that $z \in \mathcal{N}_2$. Then,*

$$z'(t) \geq \left(\frac{\psi(t)}{\varphi(t)}\left(\varphi(t)\left(z'(t)\right)^{\gamma}\right)'\Psi(t)\right)^{1/\gamma}, \tag{2.3}$$

$$z(t) \geq \mathcal{G}(t)\left(\psi(t)\left(\varphi(t)\left(z'(t)\right)^{\gamma}\right)'\right)^{1/\gamma}, \tag{2.4}$$

$$z(t) \geq \mathcal{G}(t)z'(t)\left(\frac{\varphi(t)}{\Psi(t)}\right)^{1/\gamma}, \tag{2.5}$$

and

$$\frac{z(t)}{\Omega(t)} \text{ is nonincreasing} \tag{2.6}$$

for all $t \geq t_1$ for some $t_1 \geq t_0$.

3 Main Results

The following comparison theorem is the first oscillation criterion of this study.

Theorem 3.1 *Let (2.1) holds and assume that $\phi : [t_0, \infty) \to \mathbb{R}$ is a continuous nondecreasing function satisfying $\zeta(t) \leq \phi(t) < \theta_1(t)$ for all $t \geq t_0$. If both first order retarded DEs*

$$\mathcal{U}'(t) + \mathcal{H}(t)\mathcal{Q}_*^{\beta}\big(\xi(t)\big)\mathcal{G}^{\beta}(\xi(t))\mathcal{U}^{\beta/\gamma}(\xi(t)) = 0 \tag{3.1}$$

and

$$\mathcal{V}'(t) + \mathcal{H}(t)\mathcal{Q}_{**}^{\beta}\big(\xi(t)\big)\mathcal{R}^{\beta}(t)\mathcal{V}^{\beta/\gamma}(\mu(t)) = 0 \tag{3.2}$$

are oscillatory, then Eq. (E) is oscillatory.

Proof Let $y(t)$ be a nonoscillatory solution of Eq. (E), say without loss of generality that $y(t) > 0$, $y(\theta_1(t)) > 0$, $y(\theta_2(t)) > 0$ and $y(\zeta(t)) > 0$ for all $t \geq t_1$ for some $t_1 \geq t_0$. By conclusion of Corollary 2.3, $y(t)$ is also a positive solution of Eq. (EQ) for $t \geq t_1$, and by Lemma 2.1, either $z \in \mathcal{N}_0$ or $z \in \mathcal{N}_2$ for sufficiently large t. Both cases will be dealt with separately.

First, assume that $z \in \mathcal{N}_2$. Then, we obtain from the neutral term that,

$$\begin{aligned}
&y(\theta_1(t)) \\
&= \frac{z(t)}{r_1(t)} - \frac{1}{r_1(t)}\left[\frac{z(\theta_1^{-1}(t))}{r_1(\theta_1^{-1}(t))} - \frac{y(\theta_1^{-1}(t))}{r_1(\theta_1^{-1}(t))} - \frac{r_2(\theta_1^{-1}(t))y(\theta_2\circ\theta_1^{-1}(t))}{r_1(\theta_1^{-1}(t))}\right] \\
&\quad - \frac{r_2(t)}{r_1(t)}\left[\frac{z(\theta_1^{-1}\circ\theta_2(t))}{r_1(\theta_1^{-1}\circ\theta_2(t))} - \frac{y(\theta_1^{-1}\circ\theta_2(t))}{r_1(\theta_1^{-1}\circ\theta_2(t))} - \frac{r_2(\theta_1^{-1}\circ\theta_2(t))y(\theta_2\circ\theta_1^{-1}\circ\theta_2(t))}{r_1(\theta_1^{-1}\circ\theta_2(t))}\right] \\
&\geq \frac{z(t)}{r_1(t)} - \frac{1}{r_1(t)}\frac{z(\theta_1^{-1}(t))}{r_1(\theta_1^{-1}(t))} - \frac{r_2(t)}{r_1(t)}\frac{z(\theta_1^{-1}\circ\theta_2(t))}{r_1(\theta_1^{-1}\circ\theta_2(t))}. \qquad (3.3)
\end{aligned}$$

Moreover, using the properties of the deviating arguments mentioned in condition (ii), following inequalities are provided:

$$\frac{\theta_1^{-1}(t)}{t} \geq 1 \qquad (3.4)$$

and

$$\frac{\theta_1^{-1}\circ\theta_2(t)}{t} \geq 1. \qquad (3.5)$$

In view of (2.6), it is easy to see from (3.4) and (3.5) that

$$z(\theta_1^{-1}(t)) \leq \frac{\Omega(\theta_1^{-1}(t))}{\Omega(t)}z(t) \qquad (3.6)$$

and

$$z(\theta_1^{-1}\circ\theta_2(t)) \leq \frac{\Omega(\theta_1^{-1}\circ\theta_2(t))}{\Omega(t)}z(t), \qquad (3.7)$$

respectively. Combining (3.3), (3.6) and (3.7) gives

$$y(\theta_1(t)) \geq z(t)\left[1 - \frac{1}{r_1(\theta_1^{-1}(t))}\frac{\Omega(\theta_1^{-1}(t))}{\Omega(t)} - \frac{r_2(t)}{r_1(\theta_1^{-1}\circ\theta_2(t))}\frac{\Omega(\theta_1^{-1}\circ\theta_2(t))}{\Omega(t)}\right]\frac{1}{r_1(t)},$$

which implies that

$$y(t) \geq \mathcal{Q}_*\big(\theta_1^{-1}(t)\big)z\big(\theta_1^{-1}(t)\big). \qquad (3.8)$$

Due to the fact that $\zeta(t)$ goes to infinity as $t \to \infty$, a $t_2 \geq t_1$ can be chosen such that $\zeta(t) \geq t_1$ for all $t \geq t_2$. Hence, it follows from (3.8) that

$$y\big(\zeta(t)\big) \geq \mathcal{Q}_*\big(\xi(t)\big) z\big(\xi(t)\big) \tag{3.9}$$

for $t \geq t_2$. If we combine (3.9) and (EQ), we get the following inequality:

$$\left(\psi(t)\Big(\varphi(t)\left(z'(t)\right)^{\gamma}\Big)'\right)' + \mathcal{H}(t)\mathcal{Q}_*^{\beta}\big(\xi(t)\big) z^{\beta}\big(\xi(t)\big) \leq 0, \quad t \geq t_2. \tag{3.10}$$

Therewithal, it is obvious from (2.4) that

$$z(\xi(t)) \geq \left(\psi(\xi(t))\Big(\varphi(\xi(t))\left(z'(\xi(t))\right)^{\gamma}\Big)'\right)^{1/\gamma} \mathcal{G}(\xi(t)). \tag{3.11}$$

Accordingly, in view of (3.11), the following inequality is obtained from (3.10), for $t \geq t_2$,

$$\left(\psi(t)\Big(\varphi(t)\left(z'(t)\right)^{\gamma}\Big)'\right)' + \mathcal{H}(t)\mathcal{Q}_*^{\beta}(\xi(t))\mathcal{G}^{\beta}(\xi(t))\left(\psi(\xi(t))\Big(\varphi(\xi(t))\left(z'(\xi(t))\right)^{\gamma}\Big)'\right)^{\beta/\gamma} \leq 0.$$

If one defines $\mathcal{U}(t) := \psi(t)\Big(\varphi(t)\left(z'(t)\right)^{\gamma}\Big)'$, this last inequality can be written as

$$\mathcal{U}'(t) + \mathcal{H}(t)\mathcal{Q}_*^{\beta}\big(\xi(t)\big)\mathcal{G}^{\beta}(\xi(t))\mathcal{U}^{\beta/\gamma}(\xi(t)) \leq 0, \tag{3.12}$$

and then, $\mathcal{U}(t)$ is a positive solution of (3.12). As a consequence of Corollary 1 in [21], Eq. (3.1) also has a positive solution, which contradicts our hypothesis that (3.1) is oscillatory.

Now, let's consider the case $z \in \mathcal{N}_0$. In this case, the inequalities (3.4) and (3.5) are valid again and due to $z(t)$ is strictly decreasing, it is clear that

$$z(t) \geq z(\theta_1^{-1}(t)) \tag{3.13}$$

and

$$z(t) \geq z(\theta_1^{-1} \circ \theta_2(t)). \tag{3.14}$$

Using (3.13) and (3.14) in inequality (3.3), we get

$$y(\theta_1(t)) \geq \frac{z(t)}{r_1(t)}\left[1 - \frac{1}{r_1(\theta_1^{-1}(t))} - \frac{r_2(t)}{r_1(\theta_1^{-1} \circ \theta_2(t))}\right],$$

which implies that

$$y(t) \geq \mathcal{Q}_{**}\big(\theta_1^{-1}(t)\big) z\big(\theta_1^{-1}(t)\big) \tag{3.15}$$

for all $t \geq t_1$. Hence,

$$y\big(\zeta(t)\big) \geq \mathcal{Q}_{**}\big(\xi(t)\big) z\big(\xi(t)\big) \tag{3.16}$$

for all $t \geq t_2$ for some $t_2 \geq t_1$. Combining inequality (3.16) with Eq. (EQ), we come to the conclusion that

$$\Big(\psi(t)\Big(\varphi(t)\left(z'(t)\right)^{\gamma}\Big)'\Big)' + \mathcal{H}(t)\mathcal{Q}_{**}^{\beta}\big(\xi(t)\big) z^{\beta}\big(\xi(t)\big) \leq 0 \tag{3.17}$$

for $t \geq t_2$. Furthermore, for $t \geq \eta \geq t_2$, we have

$$\varphi(t)\left(z'(t)\right)^{\gamma} - \varphi(\eta)\left(z'(\eta)\right)^{\gamma} = \int_{\eta}^{t} \frac{\psi(u)\Big(\varphi(u)\left(z'(u)\right)^{\gamma}\Big)'}{\psi(u)} \mathrm{d}u,$$

which gives

$$-\left(z'(\eta)\right)^{\gamma} \geq \psi(t)\Big(\varphi(t)\left(z'(t)\right)^{\gamma}\Big)' \left(\frac{1}{\varphi(\eta)} \int_{\eta}^{t} \frac{1}{\psi(u)} \mathrm{d}u\right).$$

Again integrating, we obtain

$$-z(t) + z(\eta) \geq \left(\int_{\eta}^{t} \left(\frac{1}{\varphi(u)} \int_{u}^{t} \frac{1}{\psi(v)} \mathrm{d}v\right)^{1/\gamma} \mathrm{d}u\right) \left(\psi(t)\Big(\varphi(t)\left(z'(t)\right)^{\gamma}\Big)'\right)^{1/\gamma}$$

and

$$z(\eta) \geq \left(\int_{\eta}^{t} \left(\frac{1}{\varphi(u)} \int_{u}^{t} \frac{1}{\psi(v)} dv\right)^{1/\gamma} du\right) \left(\psi(t)\Big(\varphi(t)\left(z'(t)\right)^{\gamma}\Big)'\right)^{1/\gamma}. \tag{3.18}$$

By virtue of $\zeta(t) \leq \phi(t) < \theta_1(t)$ for all $t \geq t_0$ and $\theta_1(t)$ is increasing, we see that $\xi(t) = \theta_1^{-1} \circ \zeta(t) \leq \mu(t) = \theta_1^{-1} \circ \phi(t) < t$. If we set $\eta = \theta_1^{-1} \circ \zeta(t)$ and $t = \theta_1^{-1} \circ \phi(t)$ in (3.18), we are led to

$$z(\xi(t)) \geq \left(\int_{\xi(t)}^{\mu(t)} \left(\frac{1}{\varphi(u)} \int_{u}^{\mu(t)} \frac{1}{\psi(v)} \mathrm{d}v\right)^{1/\gamma} \mathrm{d}u\right) \left(\psi(\mu(t))\Big(\varphi(\mu(t))\left(z'(\mu(t))\right)^{\gamma}\Big)'\right)^{1/\gamma}.$$

Substituting this last inequality into (3.17), we get

$$\Big(\psi(t)\Big(\varphi(t)\left(z'(t)\right)^{\gamma}\Big)'\Big)' + \mathcal{H}(t)\mathcal{Q}_{**}^{\beta}\big(\xi(t)\big)\mathcal{R}^{\beta}(t) \left(\psi(\mu(t))\Big(\varphi(\mu(t))\left(z'(\mu(t))\right)^{\gamma}\Big)'\right)^{\beta/\gamma} \leq 0,$$

for $t \in [t_2, \infty)$. If one denotes $\mathcal{V}(t) := \psi(t)\Big(\varphi(t)\left(z'(t)\right)^{\gamma}\Big)'$, the last inequality can be written in the form

$$\mathcal{V}'(t) + \mathcal{H}(t)\mathcal{Q}_{**}^{\beta}\big(\xi(t)\big)\mathcal{R}^{\beta}(t)\mathcal{V}^{\beta/\gamma}(\mu(t)) \leq 0, \tag{3.19}$$

and then, $\mathcal{V}(t)$ is a positive solution of (3.19). As a consequence of Corollary 1 in [21], Eq. (3.2) also has a positive solution, which contradicts our hypothesis that (3.2) is oscillatory. The proof is complete. □

One of the well-known oscillation results of [15] is that: If

$$\liminf_{t\to\infty} \int_{\delta(t)}^{t} H(\eta)\mathrm{d}\eta > \frac{1}{e}, \tag{3.20}$$

then the first-order delay DE

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)\big] + H(t)x\left[\delta(t)\right] = 0 \tag{3.21}$$

oscillates, where the functions $H, \delta \in C\left([t_0, \infty), \mathbb{R}\right)$, $\delta(t) < t$, $\lim_{t\to\infty}\delta(t) = \infty$ and $H(t)$ is nonnegative. Accordingly, the next corollary is an immediate oscillation criterion, by virtue of Theorem 3.1.

Corollary 3.2 *Let (2.1) holds, $\gamma = \beta$ and suppose that $\phi \in C\left([t_0, \infty), \mathbb{R}\right)$ is a non-decreasing function satisfying $\zeta(t) \leq \phi(t) < \theta_1(t)$ for all $t \geq t_0$. If*

$$\liminf_{t\to\infty} \int_{\xi(t)}^{t} \mathcal{H}(\eta)\mathcal{Q}_{*}^{\beta}\big(\xi(\eta)\big)\mathcal{G}^{\beta}(\xi(\eta))\mathrm{d}\eta > \frac{1}{e} \tag{3.22}$$

and

$$\liminf_{t\to\infty} \int_{\mu(t)}^{t} \mathcal{H}(\eta)\mathcal{Q}_{**}^{\beta}\big(\xi(\eta)\big)\mathcal{R}^{\beta}(\eta)\mathrm{d}\eta > \frac{1}{e}, \tag{3.23}$$

then every solution of Eq. (E) is oscillatory. □

The next corollary is a practicable oscillation criterion in the case when $\gamma > \beta$. Since its proof follows directly from inequalities (3.12), (3.19), Theorem 2 of [14] and Corollary 1 of [21], the details are skipped.

Corollary 3.3 *Let (2.1) holds, $\gamma > \beta$ and suppose that $\phi \in C([t_0, \infty), \mathbb{R})$ is a non-decreasing function satisfying $\zeta(t) \leq \phi(t) < \theta_1(t)$ for all $t \geq t_0$. If*

$$\int_T^\infty \mathcal{H}(\eta)\mathcal{Q}_*^\beta\big(\xi(\eta)\big)\mathcal{G}^\beta(\xi(\eta))\mathrm{d}\eta = \infty \tag{3.24}$$

and

$$\int_T^\infty \mathcal{H}(\eta)\mathcal{Q}_{**}^\beta\big(\xi(\eta)\big)\mathcal{R}^\beta(\eta)\mathrm{d}\eta = \infty \tag{3.25}$$

for all $\eta \geq T \geq t_0$, then Eq. (E) is oscillatory.

Lemma 3.1 [22] *Assume that $\varepsilon > 0$ is a real fixed number, λ is a ratio of two positive odd integers such that $\lambda > 1$. Then, the first order delay DE*

$$x'(t) + m(t)x^\lambda(t-\varepsilon) = 0$$

oscillates, provided that

$$\liminf_{t\to\infty}\left[\lambda^{-\frac{t}{\varepsilon}}\log\big(m(t)\big)\right] > 0,$$

where $m : [t_0, \infty) \to (0, \infty)$ is a continuous function.

For the case when $\gamma < \beta$, we can give following oscillation result for Eq. (E). In the next corollary, we suppose that $\zeta(t) = t - \ell_1$, $\phi(t) = t - \ell_2$, $\theta_1(t) = t - \ell_3$ and $\theta_2(t) = t + \ell_4$ where $\ell_1, \ell_2, \ell_3, \ell_4$ are positive real constants.

Corollary 3.4 *Let (2.1) holds, $\gamma < \beta$ and $\ell_1 \geq \ell_2 > \ell_3$. Then Eq. (E) is oscillatory provided that*

$$\liminf_{t\to\infty}\left[\left(\frac{\beta}{\gamma}\right)^{-\frac{t}{\ell_1-\ell_3}}\log\big(\mathcal{H}(t)\mathcal{Q}_*^\beta\big(t-\ell_1+\ell_3\big)\mathcal{G}^\beta\big(t-\ell_1+\ell_3\big)\big)\right] > 0 \tag{3.26}$$

and

$$\liminf_{t\to\infty}\left[\left(\frac{\beta}{\gamma}\right)^{-\frac{t}{\ell_2-\ell_3}}\log\big(\mathcal{H}(t)\mathcal{Q}_{**}^\beta\big(t-\ell_1+\ell_3\big)\mathcal{R}^\beta\big(t\big)\big)\right] > 0. \tag{3.27}$$

Remark 3.5 Since the neutral term of the Eq. (E) involves both retarded and advanced type deviating arguments, none of the oscillation criteria given in the [8, 23–25, 31] can be applicable to the (E). If we take $r_2(t) = 0$ in $z(t)$, then the results of this paper reduce to those in [25].

4 Examples and a Remark

In this final section, we give several illustrative examples to demonstrate applicability of the presented results, and a remark to indicate that further oscillation theorems can be established under assumptions of this study.

Example 4.1 If we take

- $\beta = \gamma = 1$, $f(t) = t^2$, $g(t) = t^{-1}$ and $h(t) = h_*/t^2$ with h_* is an element of positive real numbers;
- $\theta_2(t) = 2t$, $\theta_1(t) = t/2$, $\zeta(t) = t/4$, $r_1(t) = 256$ and $r_2(t) = 2$,

then the third-order mixed neutral DE (E) becomes a linear equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\left(y(t)+256y\left(\frac{t}{2}\right)+2y(2t)\right)\right)\right)+\frac{h_*}{t^2}y\left(\frac{t}{4}\right)=0, \tag{4.1}$$

for $t \geq 1$. So, $(i)-(iii)$ hold with

$$\int_{t_0}^{\infty}\frac{1}{f(\eta)}\mathrm{d}\eta=\int_1^{\infty}\eta^{-2}\mathrm{d}\eta=1<\infty \text{ and } \int_{t_0}^{\infty}\frac{1}{g^{1/\gamma}(\eta)}\mathrm{d}\eta=\int_1^{\infty}\eta\mathrm{d}\eta=\infty,$$

and Eq. (4.1) is in the semi-canonical form. With a simple verification, we see

$$\mathcal{F}(t)=\frac{1}{t},\quad \psi(t)=\varphi(t)=1 \text{ and } \mathcal{H}(t)=h_*/t^3,$$

and the transformed Eq. (EQ) in canonical form becomes

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(y(t)+256y\left(\frac{t}{2}\right)+2y(2t)\right)+\frac{h_*}{t^3}y\left(\frac{t}{4}\right)=0, \tag{4.2}$$

for $t \geq 1$. Letting $\phi(t) = \frac{t}{3}$ yields

$$\xi(t)=\frac{t}{2}<\mu(t)=\frac{2t}{3}<t \text{ and } \mathcal{R}(t)=\frac{t^2}{72} \text{ for } t\geq 1.$$

With a direct computation, we obtain

$$\Psi(t)=t-1,\quad \mathcal{G}(t)=\frac{(t-1)^2}{2},\quad \Omega(t)=(t-1)^2,$$
$$\mathcal{Q}_*(t)\geq 2.2736\times 10^{-3} \text{ and } \mathcal{Q}_{**}(t)=\frac{253}{65536}$$

for $t \geq t_1 = 2$. Thus, condition (3.22) becomes

$$\begin{aligned}
&\liminf_{t\to\infty} \int_{\xi(t)}^{t} \mathcal{H}(\eta)\mathcal{Q}_*^{\beta}\big(\xi(\eta)\big)\mathcal{G}^{\beta}(\xi(\eta))\mathrm{d}\eta \\
&\geq \liminf_{t\to\infty} \int_{t/2}^{t} \frac{h_*}{2^3\eta^3}(2.2736\times 10^{-3})(\eta-2)^2\mathrm{d}\eta \\
&= \liminf_{t\to\infty} \int_{t/2}^{t} \frac{2.2736h_*}{20^3}\left(\frac{1}{\eta}-\frac{4}{\eta^2}+\frac{4}{\eta^3}\right)\mathrm{d}\eta = \frac{2.2736h_*}{20^3}\ln 2
\end{aligned}$$

and (3.22) is satisfied provided that $h_* > \frac{20^3}{2.2736e\ln 2}$. Condition (3.23) becomes

$$\liminf_{t\to\infty} \int_{\mu(t)}^{t} \mathcal{H}(\eta)\mathcal{Q}_{**}^{\beta}\big(\xi(\eta)\big)\mathcal{R}^{\beta}(\eta)\mathrm{d}\eta = \liminf_{t\to\infty} \int_{2t/3}^{t} \frac{h_*}{\eta^3}\frac{253}{65536}\frac{\eta^2}{72}\mathrm{d}\eta = \frac{253h_*}{2^{19}3^2}\ln 3/2$$

and (3.23) is satisfied provided that $h_* > \frac{2^{19}3^2}{253e\ln 3/2}$. Hence, by Corollary 3.2, every solution of Eq. (4.1) oscillates provided that $h_* > \frac{2^{19}3^2}{253\ln 3/2}$. We note that the previous results for neutral type DEs given in [8, 25, 31] cannot be applicable to the Eq. (4.1).

Example 4.2 If we take

- $\beta = 5/7$, $\gamma = 1$, $r_1(t) = 10t$, $r_2(t) = t/64$, $f(t) = t^2$ and $g(t) = \frac{1}{t}$;
- $\zeta(t) = \frac{t}{5}$, $\theta_2(t) = 2t$, $\theta_1(t) = \frac{t}{3}$ and $h(t) = h_*/t^{\kappa}$ with $h_* > 0$ and $\kappa \leq 5/7$,

then the third-order mixed neutral DE (E) becomes a sublinear equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\left(y(t)+10ty\left(\frac{t}{3}\right)+\frac{t}{64}y(2t)\right)\right)\right)+\frac{h_*}{t^{\kappa}}y^{5/7}\left(\frac{t}{5}\right)=0, \tag{4.3}$$

for $t \geq 4$. So, $(i)-(iii)$ hold with

$$\int_{t_0}^{\infty}\frac{1}{f(\eta)}\mathrm{d}\eta = \int_4^{\infty}\eta^{-2}\mathrm{d}\eta = \frac{1}{4} < \infty \text{ and } \int_{t_0}^{\infty}\frac{1}{g^{1/\gamma}(\eta)}\mathrm{d}\eta = \int_4^{\infty}\eta\mathrm{d}\eta = \infty,$$

and Eq. (4.3) is in the semi-canonical form. Moreover, with a simple verification, we get

$$\mathcal{F}(t) = \frac{1}{t}, \quad \psi(t) = \varphi(t) = 1 \text{ and } \mathcal{H}(t) = h_*/t^{\kappa+1},$$

and the transformed equation in canonical form becomes

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(y(t)+10ty\left(\frac{t}{3}\right)+\frac{t}{64}y(2t)\right)+\frac{h_*}{t^{\kappa+1}}y^{5/7}\left(\frac{t}{5}\right)=0, \tag{4.4}$$

for $t \geq 4$. Note that $\theta_1^{-1} \circ \zeta(t) = 3t/5 < t$. Letting $\phi(t) = t/4$, we get

$$\theta_1^{-1} \circ \zeta(t) < \theta_1^{-1} \circ \phi(t) = \frac{3t}{4} < t \text{ and } \mathcal{R}(t) = \frac{9t^2}{800} \text{ for } t \geq 4.$$

With a direct computation, following resultings can be found that:

$$\Psi(t) = t - 4, \quad \mathcal{G}(t) = \frac{t^2}{2} - 4t + 8, \quad \Omega(t) = (t-4)^2,$$
$$\mathcal{Q}_*(t) \geq \frac{951t^3 - 7956t^2 + 16124t - 512}{9600t^2(t-4)^2} > 0$$
$$\text{and } \mathcal{Q}_{**}(t) = \frac{3839t - 128}{38400t^2} > 0$$

for $t \geq t_1 = 5$. Thus, condition (3.24) becomes

$$\int_T^\infty \mathcal{H}(\eta)\mathcal{Q}_*^\beta\big(\xi(\eta)\big)\mathcal{G}^\beta(\xi(\eta))\mathrm{d}\eta \geq \mathcal{E}_1 \int_{T_1}^\infty \frac{1}{\eta^{\kappa+2/7}}\mathrm{d}\eta = \infty$$

where $\mathcal{E}_1 \in \mathbb{R}^+$ and $T_1 \geq T$. Also, condition (3.25) becomes

$$\int_T^\infty \mathcal{H}(\eta)\mathcal{Q}_{**}^\beta\big(\xi(\eta)\big)\mathcal{R}^\beta(\eta)\mathrm{d}\eta \geq \mathcal{E}_2 \int_{T_1}^\infty \frac{1}{\eta^{\kappa+2/7}}\mathrm{d}\eta = \infty$$

where $\mathcal{E}_2 \in \mathbb{R}^+$ and $T_1 \geq T$. Hence, by Corollary 3.3, Eq. (4.3) oscillates provided that $\kappa \leq 5/7$. It should be pointed out that the results given in [8, 25, 31] cannot be applicable to the Eq. (4.3).

Example 4.3 With

- $\beta = 3$, $\gamma = 1$, $r_1(t) = 3t$, $r_2(t) = 1/2$ and $h(t) = t\exp(8^t)$;
- $f(t) = \varepsilon t^2$ with $0 < \varepsilon \in \mathbb{R}$, $g(t) = 1/\sigma t$ with $0 < \sigma \in \mathbb{R}$;
- $\zeta(t) = t - 4$ with $\ell_1 = 4$, $\theta_2(t) = t + 2$ with $\ell_4 = 2$, $\theta_1(t) = t - 2$ with $\ell_3 = 2$,

the third-order mixed neutral DE (E) becomes a superlinear equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon t^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sigma t}\frac{\mathrm{d}}{\mathrm{d}t}\left(y(t) + 3ty\,(t-2) + \frac{y(t+2)}{2}\right)\right)\right) + t\exp(8^t)y^3\,(t-4) = 0, \tag{4.5}$$

for $t \geq 5$. So, $(i) - (iii)$ hold with

$$\int_{t_0}^\infty \frac{1}{f(\eta)}\mathrm{d}\eta = \frac{1}{\varepsilon}\int_5^\infty \frac{1}{\eta^2}\mathrm{d}\eta = \frac{1}{5\varepsilon} < \infty \text{ and } \int_{t_0}^\infty \frac{1}{g^{1/\gamma}(\eta)}\mathrm{d}\eta = \sigma\int_5^\infty \eta\mathrm{d}\eta = \infty,$$

and Eq. (4.5) is in semi-canonical form. Moreover, with a simple verification, we are led to

$$\mathcal{F}(t)=\frac{1}{\varepsilon t},\quad \psi(t)=\frac{1}{\varepsilon},\quad \varphi(t)=\frac{\varepsilon}{\sigma} \text{ and } \mathcal{H}(t)=\frac{\exp(8^t)}{\varepsilon},$$

and the transformed Eq. (EQ) in canonical form becomes

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(y(t)+3ty\,(t-2)+\frac{1}{2}y(t+2)\right)+\frac{\sigma}{\varepsilon}\exp(8^t)y^3\,(t-4)=0, \tag{4.6}$$

for $t\geq 5$. Letting $\phi(t)=-3+t$ with $\ell_2=3$ yields

$$\xi(t)<\mu(t)=t-1<t \text{ and } \mathcal{R}(t)=\frac{\sigma}{2} \text{ for } t\geq 5.$$

With a direct computation, following results are obtained that:

$$\Psi(t)=\varepsilon(t-5),\quad \mathcal{G}(t)=\frac{\sigma}{2}\left(t^2-10t+25\right),\quad \Omega(t)=(t-5)^2,$$
$$\mathcal{Q}_*(t)=\frac{6t^4-27t^3-158t^2+453t+1126}{18t^5-72t^4-486t^3+1260t^2+3600t}>0$$
$$\text{and } \mathcal{Q}_{**}(t)=\frac{6t^2+33t+38}{18t^3+108t^2+144t}>0$$

for $t\geq t_1=6$. Since the limit inferiors of (3.26) and (3.27) diverge to infinity as $t\to\infty$, Eq. (4.5) oscillates, by Corollary 3.4. It should be noted that the results presented in [8, 25, 31] cannot be applicable to the Eq. (4.5).

Remark 4.4 Let's denote

$$\mathcal{Q}_{***}(t):=\left[1-\frac{1}{r_2(\theta_2^{-1}(t))}-\frac{r_2(t)}{r_1(\theta_1^{-1}\circ\theta_2(t))}\frac{\Omega(\theta_1^{-1}\circ\theta_2(t))}{\Omega(t)}\right]\frac{1}{r_1(t)},$$

where θ_1^{-1} and θ_2^{-1} are inverse functions of θ_1 and θ_2, respectively. If the function $\mathcal{Q}_{***}(t)$ is substituted instead of the function $\mathcal{Q}_*(t)$ in Theorem 3.1 and Corollaries 3.2-3.4, then some new oscillation results can be established for Eq. (E), provided that $\mathcal{Q}_{***}(t)$ is nonnegative, $r_1(t)\geq 1$, $r_2(t)\geq 1$, $r_1(t)\not\equiv 1$ and $r_2(t)\not\equiv 1$ for sufficiently large t, in condition (i).

5 Conclusion

In this paper, we have considered a class of third-order nonlinear semi-canonical functional DEs whose oscillatory behaviors have not been performed before in the case when the neutral term is of mixed type. Under certain conditions, we have shown that a semi-canonical equation can be transformed to an equivalent equation in canonical

form. So it is interesting to observe that the results presented in this work could be applied to both types of equations. In view of this connection, our results are new and different from others reported in the literature.

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