Monatshefte für Mathematik (2021) 194:85–104
https://doi.org/10.1007/s00605-020-01469-6

ORIGINAL RESEARCH

Oscillation of damped second-order linear mixed neutral differential equations

John R. Graef[1] · Orhan Özdemir[2] · Adil Kaymaz[2] · Ercan Tunç[2]

Received: 25 March 2019 / Accepted: 17 September 2020 / Published online: 25 September 2020
© Springer-Verlag GmbH Austria, part of Springer Nature 2020

Abstract
This article concerns the oscillatory behavior of solutions to damped second-order linear functional differential equations with a mixed neutral term. The authors present new oscillation criteria that improve and extend some existing ones in the literature. Examples to illustrate the main results are included.

Keywords Oscillation · Neutral differential equations of mixed type · Second-order · Damping term

Mathematics Subject Classification 34C10 · 34K11 · 34K40

1 Introduction

This paper is concerned with oscillatory behavior of solutions of the damped second-order linear neutral differential equation of mixed type

$$\left(r(t)z'(t)\right)' + p(t)z'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

Communicated by Adrian Constantin.

✉ John R. Graef
John-Graef@utc.edu

Orhan Özdemir
orhanozdemir37@yahoo.com

Adil Kaymaz
adilkaymaz@gmail.com

Ercan Tunç
ercantunc72@yahoo.com

[1] Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA

[2] Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, 60240 Tokat, Turkey

where

$$z(t) = x(t) + p_1(t)x(\tau_1(t)) + p_2(t)x(\tau_2(t)).$$

We will make use of the following conditions:

(i) $p, q, r : [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions with $p(t) \geq 0$, $r(t) > 0$, $q(t) \geq 0$ with $q(t)$ not identically zero for large t, and

$$\int_{t_0}^{\infty} \frac{1}{r(t)} \exp\left(-\int_{t_0}^{t} p(s)/r(s)ds\right) dt = \infty; \tag{1.2}$$

(ii) $\tau_1, \tau_2, \sigma : [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions such that τ_1 and τ_2 are strictly increasing, $\tau_1(t) < t$, $\tau_2(t) > t$, and $\lim_{t\to\infty} \tau_1(t) = \lim_{t\to\infty} \tau_2(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

and, either

(iii$_a$) $p_1, p_2 : [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions such that $p_1(t) > 0$, and $p_2(t) \geq 1$ with $p_2(t) \not\equiv 1$ for large t;

or

(iii$_b$) $p_1, p_2 : [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions such that $p_2(t) > 0$, and $p_1(t) \geq 1$ with $p_1(t) \not\equiv 1$ for large t.

By a *solution* of Eq. (1.1), we mean a function $x \in C([t_x, \infty), \mathbb{R})$ for some $t_x \geq t_0$ that has the properties $z \in C^1([t_x, \infty), \mathbb{R})$, $rz' \in C^1([t_x, \infty), \mathbb{R})$, and satisfies (1.1) on $[t_x, \infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_x, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \text{ for any } T_1 \geq t_x;$$

moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, i.e., for any $t_1 \in [t_x, \infty)$ there exists $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1.1) itself is termed oscillatory if all its solutions are oscillatory.

The oscillatory behavior of solutions of second order neutral differential equations with a delayed or advanced argument has been widely studied by many authors utilizing different methods; see, for example, [1,2,5–8,10–15,23] and the references therein. However, compared to second order neutral differential equations with delayed or advanced arguments, oscillation results for second order neutral differential equations with mixed arguments are scarce; and moreover, the results obtained are concerned with the equations without a damping term; see the papers [3,4,9,16–19,21,22,24] and the references therein as examples of recent results on this topic. On the other hand, the types of equations considered in the relevant literature are generally as follows.

Using a comparison technique, Li [17] and Li et al. [18] studied the oscillation of solutions of the second order neutral differential equation with constant mixed

arguments

$$\left(r(t)z'(t)\right)' + q_1(t)x(t-\sigma_3) + q_2(t)x(t+\sigma_4) = 0, \tag{1.3}$$

where $z(t) = x(t)+p_1(t)x(t-\sigma_1)+p_2(t)x(t+\sigma_2)$. Ji et al. [9] considered the second-order half-linear neutral dynamic equation with mixed time dependent arguments

$$\left(r(t)\left(z^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q_1(t)x^{\gamma}(\tau_1(t)) + q_2(t)x^{\gamma}(\tau_2(t)) = 0, \tag{1.4}$$

where $z(t) = x(t) + p_1(t)x(\eta_1(t)) + p_2(t)x(\eta_2(t))$.

Qi and Yu [19], and Zhang et al. [24] studied the oscillation of solutions of the second order neutral differential equation with constant mixed arguments and distributed deviating arguments

$$\left(r(t)\left(z'(t)\right)^{\gamma}\right)' + \int_a^b q_1(t,\xi)x^{\gamma}(t-\xi)d\xi + \int_a^b q_2(t,\xi)x^{\gamma}(t+\xi)d\xi = 0, \tag{1.5}$$

where $z(t) = x(t) + p_1(t)x(t-\sigma_1) + p_2(t)x(t+\sigma_2)$.

Li et al. [16] studied the oscillation of solutions of the second order half-linear neutral differential equation with mixed arguments

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + q(t)\left|x(\sigma(t))\right|^{\alpha-1}x(\sigma(t)) = 0 \tag{1.6}$$

where $z(t) = x(t) + \Sigma_{i=1}^{m} p_i(t)x(\tau_i(t))$.

Arul and Shobha [4] investigated the oscillation of solutions of the second order neutral differential equation with mixed constant arguments

$$\left(r(t)z'(t)\right)' + q(t)f(x(\sigma(t))) = 0, \tag{1.7}$$

where $z(t) = x(t) + p_1(t)x(t-\tau) + p_2(t)x(t+\delta)$.

The oscillation of solutions of the second-order nonlinear neutral dynamic equation with mixed arguments

$$\left(r(t)\left(z^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + f(t,x(\tau_1(t))) + g(t,x(\tau_2(t))) = 0, \tag{1.8}$$

where $z(t) = x(t) + p_1(t)x(\eta_1(t)) + p_2(t)x(\eta_2(t))$, was considered by Agwa et al. [3].

Thandapani and Rama [21] and Thandapani et al. [22] studied the oscillation of solutions of the second order neutral differential equation with mixed arguments

$$\left(r(t)\left[z^{\alpha}(t)\right]'\right)' + q_1(t)x^{\beta}(t-\tau_1) + q_2(t)x^{\gamma}(t+\tau_2) = 0, \tag{1.9}$$

where $z(t) = x(t) + p_1(t)x(t-\sigma_1) + p_2(t)x(t+\sigma_2)$.

One thing that is important to point out is that in all the papers mentioned above, the coefficient functions $p_1(t)$ and $p_2(t)$ are positive and bounded above.

Motivated by the papers mentioned above, here we wish to establish some new oscillation criteria for Eq. (1.1). We note that the results presented in this paper are new even for constant delays such as $\tau_1(t) = t - \sigma_1$, $\tau_2(t) = t + \sigma_2$, and $\sigma(t) = t \pm \sigma_3$ with $\sigma_i(t) > 0$ for $i = 1, 2, 3$. It should also be noted that the results in the present paper can easily be extended to the damped form of Eqs. (1.3)–(1.9) as well as to more general mixed neutral differential equations with a damping term. For these reasons, it is hoped that the present paper not only fills a gap in oscillation theory for damped second order mixed neutral differential equations, but also inspires additional work in this area.

2 Oscillation results for (1.1) in the case where (iii$_a$) holds

In this section, we establish some new criteria for the oscillation of Eq. (1.1) in the cases where $\sigma(t) \le \tau_2(t)$ and $\sigma(t) \ge \tau_2(t)$, respectively. For notational purposes, we let

$$R(t, t_1) = \int_{t_1}^{t} \frac{1}{r(s)} ds \quad \text{for } t \ge t_1 \ge t_0,$$

and for any continuous function $u(t)$ we set $u_+(t) = \max\{u(t), 0\}$. In addition, we define $\psi : [t_0, \infty) \to \mathbb{R}$ by

$$\psi(t) := \frac{1}{p_2(\tau_2^{-1}(t))} \left[1 - \frac{1}{p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} - \frac{p_1(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))} \right],$$

where τ_2^{-1} is the inverse function of τ_2. Throughout this section we assume that $\psi(t) > 0$ for all sufficiently large t.

We begin with the following lemma.

Lemma 2.1 *Assume that*

$$z(t) > 0, \quad z'(t) > 0, \quad \text{and} \quad \left(r(t)z'(t)\right)' \le 0 \tag{2.1}$$

for $t \ge t_1$, for some $t_1 \ge t_0$. Then

$$z(t) \ge r(t)z'(t)R(t, t_1) \quad \text{for } t \ge t_1. \tag{2.2}$$

Also, there exists $t_2 > t_1$ such that

$$\left(\frac{z(t)}{R(t, t_1)} \right)' \le 0 \quad \text{for } t \ge t_2. \tag{2.3}$$

Proof Assume that (2.1) holds for $t \geq t_1$ for some $t_1 \geq t_0$. Then we see that

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{r(s)z'(s)}{r(s)} ds \geq r(t)z'(t) \int_{t_1}^{t} \frac{1}{r(s)} ds = r(t)z'(t)R(t, t_1),$$

i.e., (2.2) holds. Next, from (2.2), we have for all $t \geq t_2 := t_1 + 1$ that

$$\left(\frac{z(t)}{R(t, t_1)} \right)' = \frac{1/r(t) \left[r(t)z'(t)R(t, t_1) - z(t) \right]}{(R(t, t_1))^2} \leq 0,$$

i.e., (2.3) holds and completes the proof. □

Theorem 2.2 *Let conditions* (i)–(iii_a) *hold and* $\sigma(t) \leq \tau_2(t)$. *If there exists a positive function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t \to \infty} \int_{T}^{t} \left[\eta(s)q(s)\psi(\sigma(s)) \frac{R\left(\tau_2^{-1}(\sigma(s)), t_1\right)}{R(s, t_1)} - \frac{\eta(s)r(s)\xi^2(s)}{4} \right] ds = \infty \tag{2.4}$$

for all sufficiently large $t_1 \in [t_0, \infty)$ *and all* $T > t_1$, *where*

$$\xi(t) = \frac{\eta'(t)r(t) - \eta(t)p(t)}{\eta(t)r(t)}, \tag{2.5}$$

then Eq. (1.1) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. (The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper). It follows from (1.1) that

$$\left(r(t)z'(t)\right)' + p(t)z'(t) = -q(t)x(\sigma(t)) \leq 0,$$

i.e.,

$$\left(r(t)z'(t)\right)' + p(t)z'(t) \leq 0 \quad \text{for } t \geq t_1.$$

Letting $y(t) = r(t)z'(t)$, the last inequality becomes

$$y'(t) + \frac{p(t)}{r(t)} y(t) \leq 0,$$

which implies

$$\left(\exp\left(\int_{t_1}^{t} p(s)/r(s) ds \right) y(t) \right)' \leq 0 \quad \text{for } t \geq t_1.$$

Thus, $\exp\left(\int_{t_1}^{t} p(s)/r(s)ds\right) y(t)$ is nonincreasing and eventually does not change its sign, say on $[t_2, \infty)$ for some $t_2 \geq t_1$. Therefore, $z'(t)$ eventually has a fixed sign on $[t_2, \infty)$, and so we have two cases to consider: (I) $z'(t) > 0$ for $t \geq t_2$ or (II) $z'(t) < 0$ for $t \geq t_2$.

First, we consider case (I). Since $z'(t) > 0$ for $t \geq t_2$, from (1.1) we see that $\left(r(t)z'(t)\right)' \leq 0$ for $t \geq t_2$, and so, in view of Lemma 2.1, we have that (2.3) holds for $t \geq t_3$ for some $t_3 \in (t_2, \infty)$, i.e., $z(t)/R(t, t_1)$ is nonincreasing for $t \geq t_3$. In view of the definition of $z(t)$, we obtain (also see inequality (8.6) in [1])

$$\begin{aligned}
x(t) &= \frac{1}{p_2(\tau_2^{-1}(t))}\left[z(\tau_2^{-1}(t)) - x(\tau_2^{-1}(t)) - p_1(\tau_2^{-1}(t))x(\tau_1(\tau_2^{-1}(t)))\right] \\
&= \frac{z(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))} - \frac{z(\tau_2^{-1}(\tau_2^{-1}(t))) - x(\tau_2^{-1}(\tau_2^{-1}(t)))}{p_2(\tau_2^{-1}(t))p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} \\
&\quad - \frac{p_1(\tau_2^{-1}(\tau_2^{-1}(t)))x(\tau_1(\tau_2^{-1}(\tau_2^{-1}(t))))}{p_2(\tau_2^{-1}(t))p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} \\
&\quad - \frac{p_1(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))}\left[\frac{z(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))) - x(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}\right. \\
&\quad \left. - \frac{p_1(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))x(\tau_1(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}\right] \\
&\geq \frac{z(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))} - \frac{z(\tau_2^{-1}(\tau_2^{-1}(t)))}{p_2(\tau_2^{-1}(t))p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} \\
&\quad - \frac{p_1(\tau_2^{-1}(t))z(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}{p_2(\tau_2^{-1}(t))p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}.
\end{aligned} \tag{2.6}$$

Using the fact that z, τ_1 and τ_2 are strictly increasing, and taking $\tau_1(t) < t < \tau_2(t)$ into account, we have

$$z(\tau_2^{-1}(\tau_2^{-1}(t))) < z(\tau_2^{-1}(t)) \tag{2.7}$$

and

$$z(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))) < z(\tau_2^{-1}(t)). \tag{2.8}$$

Using (2.7) and (2.8) in (2.6) gives

$$x(t) \geq \frac{1}{p_2(\tau_2^{-1}(t))}\left[1 - \frac{1}{p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} - \frac{p_1(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}\right] z(\tau_2^{-1}(t))$$

or

$$x(t) \geq \psi(t)z\left(\tau_2^{-1}(t)\right) \quad \text{for } t \geq t_3. \tag{2.9}$$

Since $\lim_{t\to\infty}\sigma(t)=\infty$, we can choose $t_4\geq t_3$ such that $\sigma(t)\geq t_3$ for all $t\geq t_4$. Thus, from (2.9) we obtain

$$x(\sigma(t))\geq\psi(\sigma(t))z\left(\tau_2^{-1}(\sigma(t))\right)\quad\text{for } t\geq t_4. \tag{2.10}$$

Using (2.10) in (1.1) gives

$$\left(r(t)z'(t)\right)'+p(t)z'(t)+q(t)\psi(\sigma(t))z\left(\tau_2^{-1}(\sigma(t))\right)\leq 0\quad\text{for } t\geq t_4. \tag{2.11}$$

Define the function $w(t)$ by the Riccati type substitution

$$w(t):=\eta(t)\frac{r(t)z'(t)}{z(t)}\quad\text{for } t\geq t_4. \tag{2.12}$$

Clearly, $w(t)>0$, and from (2.11) and (2.12), we see that

$$\begin{aligned}w'(t)&=\frac{\eta'(t)}{\eta(t)}w(t)+\eta(t)\left[\frac{\left(r(t)z'(t)\right)'z(t)-r(t)(z'(t))^2}{z^2(t)}\right]\\&\leq\frac{\eta'(t)}{\eta(t)}w(t)+\frac{\eta(t)}{z(t)}\left[-p(t)z'(t)-q(t)\psi(\sigma(t))z\left(\tau_2^{-1}(\sigma(t))\right)\right]-\frac{w^2(t)}{\eta(t)r(t)}\\&=\frac{\eta'(t)}{\eta(t)}w(t)-\frac{p(t)}{r(t)}w(t)-\eta(t)q(t)\psi(\sigma(t))\frac{z\left(\tau_2^{-1}(\sigma(t))\right)}{z(t)}-\frac{w^2(t)}{\eta(t)r(t)}.\end{aligned} \tag{2.13}$$

In view of (2.5), inequality (2.13) takes the form

$$w'(t)\leq\xi(t)w(t)-\eta(t)q(t)\psi(\sigma(t))\frac{z\left(\tau_2^{-1}(\sigma(t))\right)}{z(t)}-\frac{1}{\eta(t)r(t)}w^2(t). \tag{2.14}$$

Using the fact $z(t)/R(t,t_1)$ is nonincreasing on $[t_4,\infty)\subseteq[t_3,\infty)$, and noting that $\sigma(t)\leq\tau_2(t)$ implies $\tau_2^{-1}(\sigma(t))\leq t$, we obtain

$$\frac{z\left(\tau_2^{-1}(\sigma(t))\right)}{z(t)}\geq\frac{R\left(\tau_2^{-1}(\sigma(t)),t_1\right)}{R(t,t_1)}. \tag{2.15}$$

Substituting (2.15) into (2.14) gives

$$w'(t)\leq\xi(t)w(t)-\eta(t)q(t)\psi(\sigma(t))\frac{R\left(\tau_2^{-1}(\sigma(t)),t_1\right)}{R(t,t_1)}-\frac{1}{\eta(t)r(t)}w^2(t) \tag{2.16}$$

for $t \geq t_4$. Completing the square with respect to w, it follows from (2.16) that

$$w'(t) \leq -\eta(t)q(t)\psi(\sigma(t))\frac{R\left(\tau_2^{-1}(\sigma(t)), t_1\right)}{R(t, t_1)} + \frac{\eta(t)r(t)\xi^2(t)}{4} \quad \text{for } t \geq t_4. \tag{2.17}$$

Integrating (2.17) from t_4 to t, we see that

$$\int_{t_4}^{t} \left[\eta(s)q(s)\psi(\sigma(s))\frac{R\left(\tau_2^{-1}(\sigma(s)), t_1\right)}{R(s, t_1)} - \frac{\eta(s)r(s)\xi^2(s)}{4}\right] ds \leq -w(t) + w(t_4) < w(t_4),$$

which contradicts (2.4).

Next, we consider case (II). Letting $u(t) = -r(t)z'(t) > 0$, it follows from (1.1) that

$$u'(t) + \frac{p(t)}{r(t)}u(t) \geq 0 \quad \text{for } t \geq t_2.$$

Integrating this relation from t_2 to t, we obtain

$$u(t) \geq u(t_2)\exp\left(-\int_{t_2}^{t} p(s)/r(s)ds\right),$$

from which we see that

$$z'(t) \leq r(t_2)z'(t_2)\frac{1}{r(t)}\exp\left(-\int_{t_2}^{t} p(s)/r(s)ds\right). \tag{2.18}$$

Integrating (2.18) from t_2 to t and taking (1.2) into account, we obtain

$$z(t) \leq z(t_2) + r(t_2)z'(t_2)\int_{t_2}^{t}\frac{1}{r(s)}\exp\left(-\int_{t_2}^{s} p(u)/r(u)du\right)ds \to -\infty \text{ as } t \to \infty,$$

which contradicts the positivity of $z(t)$. This proves the theorem. □

Theorem 2.3 *Let conditions* (i)–(iii_a) *hold and* $\sigma(t) \leq \tau_2(t)$. *If there exists a positive function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{T}^{t}\left[\eta(s)q(s)\psi(\sigma(s))\frac{R\left(\tau_2^{-1}(\sigma(s)), t_1\right)}{R(s, t_1)} - \frac{\eta'_+(s)}{R(s, t_1)}\right]ds = \infty \tag{2.19}$$

for all sufficiently large $t_1 \in [t_0, \infty)$ *and for all* $T > t_1$, *then Eq.* (1.1) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \in [t_0, \infty)$. Proceeding as in the proof of Theorem 2.2, we again have the two cases to consider: (I) $z'(t) > 0$ for $t \geq t_2$ or (II) $z'(t) < 0$ for $t \geq t_2$. If case (II) holds, proceeding exactly as in the proof of Theorem 2.2, we obtain a contradiction to the positivity of z.

Next, assume that case (I) holds. Proceeding as in the proof of Theorem 2.2, we again arrive at (2.14) and (2.15) for $t \geq t_4 \geq t_3$. Since $z(t) > 0$, $z'(t) > 0$, and $\left(r(t)z'(t)\right)' \leq 0$, in view of Lemma 2.1, we see that (2.2) holds for $t \geq t_4$. From (2.2), (2.5), (2.12), and the fact that $w(t) > 0$, inequality (2.14) can be written as

$$w'(t) \leq \frac{\eta'_+(t)}{R(t,t_1)} - \eta(t)q(t)\psi(\sigma(t))\frac{z\left(\tau_2^{-1}(\sigma(t))\right)}{z(t)}. \tag{2.20}$$

Using (2.15) in (2.20) gives

$$w'(t) \leq \frac{\eta'_+(t)}{R(t,t_1)} - \eta(t)q(t)\psi(\sigma(t))\frac{R\left(\tau_2^{-1}(\sigma(t)),t_1\right)}{R(t,t_1)} \tag{2.21}$$

for $t \geq t_4$. Integrating (2.21) from t_4 to t yields

$$\int_{t_4}^{t}\left[\eta(s)q(s)\psi(\sigma(s))\frac{R\left(\tau_2^{-1}(\sigma(s)),t_1\right)}{R(s,t_1)} - \frac{\eta'_+(s)}{R(s,t_1)}\right]ds \leq -w(t) + w(t_4) < w(t_4),$$

which contradicts (2.19) and completes the proof of the theorem. □

In the following theorem, we establish a new oscillation criterion for (1.1) by using the integral averaging technique due to Philos [20]. In order to present our theorem, following Philos [20], we first introduce the function class $\mathcal{P}$. Namely, let $D_0 = \{(t,s) : t > s \geq t_0\}$ and $D = \{(t,s) : t \geq s \geq t_0\}$. We say that the function $H \in C(D, \mathbb{R})$ belongs to the class $\mathcal{P}$, denoted by $H \in \mathcal{P}$, if

($\mathcal{P}_1$) $H(t,t) = 0$ for $t \geq t_0$, and $H(t,s) > 0$ on $(t,s) \in D_0$;
($\mathcal{P}_2$) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable.

Theorem 2.4 *Assume that conditions* (i)–(iii_a) *hold,* $\sigma(t) \leq \tau_2(t)$, *and let* $h, H : D \to \mathbb{R}$ *be continuous functions such that* H *belongs to the class* $\mathcal{P}$ *and*

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in D_0. \tag{2.22}$$

Set

$$\Psi(t,s) = -h(t,s) + \sqrt{H(t,s)}\xi(s), \tag{2.23}$$

where ξ is given in (2.5). If there exists a positive function $\eta \in C^1([t_0, \infty), \mathbb{R})$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[M(t,s) - \frac{\eta(s) r(s) \Psi^2(t,s)}{4} \right] ds = \infty \tag{2.24}$$

for all sufficiently large $t_1 \in [t_0, \infty)$ and for all $T > t_1$, where

$$M(t,s) = H(t,s)\eta(s)q(s)\psi(\sigma(s)) \frac{R\left(\tau_2^{-1}(\sigma(s)), t_1\right)}{R(s,t_1)},$$

then every solution of (1.1) is oscillatory.

Proof Let $x(t)$ be a nonoscillatory solution of (1.1) with $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.2, we again have the two cases: (I) $z'(t) > 0$ for $t \geq t_2$ or (II) $z'(t) < 0$ for $t \geq t_2$. If case (II) holds, as in the proof of Theorem 2.2, we contradict the positivity of $z(t)$.

If case (I) holds, then, as in the proof of Theorem 2.2, we again arrive at (2.16) for $t \geq t_4$. From (2.16), it follows that

$$\begin{aligned} \int_{t_4}^t M(t,s)ds \leq & -\int_{t_4}^t H(t,s)w'(s)ds + \int_{t_4}^t H(t,s)\xi(s)w(s)ds \\ & - \int_{t_4}^t H(t,s)\frac{1}{\eta(s)r(s)}w^2(s)ds. \end{aligned} \tag{2.25}$$

An integrating by parts yields

$$\begin{aligned} \int_{t_4}^t H(t,s)w'(s)ds &= H(t,s)w(s)\Big|_{t_4}^t - \int_{t_4}^t \frac{\partial H}{\partial s}(t,s)w(s)ds \\ &= -H(t,t_4)w(t_4) - \int_{t_4}^t \frac{\partial H}{\partial s}(t,s)w(s)ds. \end{aligned} \tag{2.26}$$

Substituting (2.26) into (2.25) gives

$$\begin{aligned} \int_{t_4}^t M(t,s)ds \leq & H(t,t_4)w(t_4) - \int_{t_4}^t H(t,s)\frac{1}{\eta(s)r(s)}w^2(s)ds \\ & + \int_{t_4}^t \left[\frac{\partial H}{\partial s}(t,s) + H(t,s)\xi(s) \right] w(s)ds. \end{aligned}$$

In view of (2.22), the last inequality takes the form

$$\int_{t_4}^{t} M(t,s)ds \le H(t,t_4)w(t_4) - \int_{t_4}^{t} H(t,s)\frac{1}{\eta(s)r(s)}w^2(s)ds + \int_{t_4}^{t}\left[-h(t,s)\sqrt{H(t,s)} + H(t,s)\xi(s)\right]w(s)ds.$$

Completing the square, we see that

$$\int_{t_4}^{t} M(t,s)ds \le H(t,t_4)w(t_4) + \frac{1}{4}\int_{t_4}^{t}\eta(s)r(s)\Psi^2(t,s)ds,$$

and so for $t > t_4 \ge t_3$,

$$\frac{1}{H(t,t_4)}\int_{t_4}^{t}\left[M(t,s) - \frac{\eta(s)r(s)\Psi^2(t,s)}{4}\right]ds \le w(t_4).$$

Taking the lim sup as $t \to \infty$ in the last inequality, we obtain a contradiction to (2.24). □

From Theorem 2.4, we immediately have the following oscillation criterion.

Corollary 2.5 *Suppose that all conditions of Theorem 2.4 are satisfied with (2.24) replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t} M(t,s)ds = \infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\eta(s)r(s)\Psi^2(t,s)ds < \infty.$$

Then Eq. (1.1) is oscillatory.

Theorem 2.6 *Let conditions (i)–(iii_a) hold and $\sigma(t) \ge \tau_2(t)$. If there exists a positive function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\limsup_{t\to\infty}\int_{T}^{t}\left[\eta(s)q(s)\psi(\sigma(s)) - \frac{\eta(s)r(s)\xi^2(s)}{4}\right]ds = \infty \tag{2.27}$$

for all sufficiently large $T \in (t_0,\infty)$, where ξ is given in (2.5), then Eq. (1.1) is oscillatory.

Proof Let $x(t)$ be a nonoscillatory solution of (1.1) with $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1 \ge t_0$. As in the proof of Theorem 2.2, we

again have two cases: (I) $z'(t) > 0$ for $t \geq t_2$ or (II) $z'(t) < 0$ for $t \geq t_2$. If case (II) holds, then again we obtain a contradiction to the positivity of z.

Next, assume that case (I) holds. Proceeding as in the proof of Theorem 2.2, we again arrive at (2.14) for $t \geq t_4$. Using the fact that τ_2 is strictly increasing and taking $\sigma(t) \geq \tau_2(t)$ into account, we see that

$$\tau_2^{-1}(\sigma(t)) \geq t;$$

from this and the fact that z is increasing, we obtain

$$\frac{z\left(\tau_2^{-1}(\sigma(t))\right)}{z(t)} \geq 1. \tag{2.28}$$

Using (2.28) in (2.14) yields

$$w'(t) \leq \xi(t)w(t) - \eta(t)q(t)\psi(\sigma(t)) - \frac{1}{\eta(t)r(t)}w^2(t). \tag{2.29}$$

The remainder of the proof is similar to the first part of the proof of Theorem 2.2, and so we omit the details. □

Theorem 2.7 *Let conditions* (i)–(iii_a) *hold and* $\sigma(t) \geq \tau_2(t)$. *If there exists a positive function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left[\eta(s)q(s)\psi(\sigma(s) - \frac{\eta'_+(s)}{R(s, t_1)}\right] ds = \infty \tag{2.30}$$

for all sufficiently large $t_1 \in [t_0, \infty)$ *and all* $T > t_1$, *then Eq.* (1.1) *is oscillatory.*

Proof The proof follows from (2.20), (2.28), and Theorem 2.3, so we omit the details. □

Theorem 2.8 *Assume that conditions* (i)–(iii_a) *hold,* $\sigma(t) \geq \tau_2(t)$, *and let* $h, H : D \to \mathbb{R}$ *be continuous functions such that* H *belongs to the class* $\mathcal{P}$ *and* (2.22) *holds. If there exists a positive function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_T^t \left[H(t, s)\eta(s)q(s)\psi(\sigma(s)) - \frac{\eta(s)r(s)\Psi^2(t, s)}{4}\right] ds = \infty \tag{2.31}$$

holds for all sufficiently large $T \in (t_0, \infty)$, *where* $\Psi(t, s)$ *is given in* (2.23), *then Eq.* (1.1) *is oscillatory.*

Proof The proof follows from (2.14), (2.28) and Theorem 2.4, and so we omit the details. □

Corollary 2.9 *Suppose that all conditions of Theorem 2.8 are satisfied with (2.31) replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\eta(s)q(s)\psi(\sigma(s))ds=\infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \eta(s)r(s)\Psi^2(t,s)ds<\infty.$$

Then Eq. (1.1) is oscillatory.

3 Oscillation results for (1.1) in the case where (iii_b) holds

In this section, we establish some new criteria for the oscillation of Eq. (1.1) in the cases where $\sigma(t)\le\tau_1(t)$ and $\sigma(t)\ge\tau_1(t)$, respectively, under the covering assumption that (iii_b) holds. For notational purposes, we let

$$\phi(t):=\frac{1}{p_1(\tau_1^{-1}(t))}\Bigg[1-\frac{1}{p_1(\tau_1^{-1}(\tau_1^{-1}(t)))}\frac{R\left(\tau_1^{-1}(\tau_1^{-1}(t)),t_1\right)}{R\left(\tau_1^{-1}(t),t_1\right)}$$
$$-\frac{p_2(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))}\frac{R\left(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))),t_1\right)}{R\left(\tau_1^{-1}(t),t_1\right)}\Bigg],$$

where τ_1^{-1} is the inverse function of τ_1. Throughout this section we assume that $\phi(t)>0$ for all sufficiently large t.

Theorem 3.1 *Let conditions (i), (ii), and (iii_b) hold and $\sigma(t)\le\tau_1(t)$. If there exists a positive function $\eta\in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\limsup_{t\to\infty}\int_T^t\left[\eta(s)q(s)\phi(\sigma(s))\frac{R\left(\tau_1^{-1}(\sigma(s)),t_1\right)}{R(s,t_1)}-\frac{\eta(s)r(s)\xi^2(s)}{4}\right]ds=\infty \tag{3.1}$$

for all sufficiently large $t_1\in[t_0,\infty)$ and all $T>t_1$, where ξ is given in (2.5), then Eq. (1.1) is oscillatory.

Proof Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t)>0$, $x(\tau_1(t))>0$, $x(\tau_2(t))>0$, and $x(\sigma(t))>0$ for $t\ge t_1\ge t_0$. Proceeding as in the proof of Theorem 2.2, we again consider the two cases: (I) $z'(t)>0$ for $t\ge t_2$ or (II) $z'(t)<0$ for $t\ge t_2$. If case (II) holds, a proof like the corresponding part of the proof of Theorem 2.2 yields a contradiction to the positivity of z.

Next, assume that case (I) holds. Following a similar procedure to the proof of Theorem 2.2, from the definition of $z(t)$, we obtain the following inequality

$$x(t) \geq \frac{z(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))} - \frac{z(\tau_1^{-1}(\tau_1^{-1}(t)))}{p_1(\tau_1^{-1}(t))p_1(\tau_1^{-1}(\tau_1^{-1}(t)))} - \frac{p_2(\tau_1^{-1}(t))z(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))}{p_1(\tau_1^{-1}(t))p_1(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))}, \tag{3.2}$$

which is analogous to (2.6). Using the fact that τ_1 and τ_2 are strictly increasing, and noting that $\tau_1(t) < t < \tau_2(t)$, we have

$$\tau_1^{-1}(\tau_1^{-1}(t)) > \tau_1^{-1}(t) \tag{3.3}$$

and

$$\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))) > \tau_1^{-1}(t). \tag{3.4}$$

Again using the fact that $z(t)/R(t, t_1)$ is nonincreasing for $t \geq t_3$ for some $t_3 \in (t_2, \infty)$, it follows from (3.3) and (3.4) that

$$\frac{R\left(\tau_1^{-1}(\tau_1^{-1}(t)), t_1\right) z(\tau_1^{-1}(t))}{R\left(\tau_1^{-1}(t), t_1\right)} \geq z(\tau_1^{-1}(\tau_1^{-1}(t))) \tag{3.5}$$

and

$$\frac{R\left(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))), t_1\right) z(\tau_1^{-1}(t))}{R\left(\tau_1^{-1}(t), t_1\right)} \geq z(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t)))) \tag{3.6}$$

respectively, for $t \geq t_3$. Using (3.5) and (3.6) in (3.2) yields

$$x(t) \geq \phi(t) z(\tau_1^{-1}(t)) \tag{3.7}$$

for $t \geq t_3$. Since $\lim_{t\to\infty} \sigma(t) = \infty$, we can choose $t_4 \geq t_3$ such that $\sigma(t) \geq t_3$ for all $t \geq t_4$. Thus, from (3.7) we obtain

$$x(\sigma(t)) \geq \phi(\sigma(t)) z\left(\tau_1^{-1}(\sigma(t))\right) \quad \text{for } t \geq t_4. \tag{3.8}$$

Substituting (3.8) in (1.1) gives

$$\left(r(t)z'(t)\right)' + p(t)z'(t) + q(t)\phi(\sigma(t))z\left(\tau_1^{-1}(\sigma(t))\right) \leq 0 \quad \text{for } t \geq t_4. \tag{3.9}$$

Again define the Riccati substitution w by (2.12). Then, it follows from (2.12) and (3.9) that

$$w'(t) \le \xi(t)w(t) - \eta(t)q(t)\phi(\sigma(t))\frac{z\left(\tau_1^{-1}(\sigma(t))\right)}{z(t)} - \frac{1}{\eta(t)r(t)}w^2(t). \quad (3.10)$$

Since $z(t)/R(t,t_1)$ is nonincreasing on $[t_4,\infty) \subseteq [t_3,\infty)$, and noting that $\sigma(t) \le \tau_1(t)$ implies $\tau_1^{-1}(\sigma(t)) \le t$, we obtain

$$\frac{z\left(\tau_1^{-1}(\sigma(t))\right)}{z(t)} \ge \frac{R\left(\tau_1^{-1}(\sigma(t)),t_1\right)}{R(t,t_1)}. \quad (3.11)$$

Substituting (3.11) into (3.10) gives

$$w'(t) \le \xi(t)w(t) - \eta(t)q(t)\phi(\sigma(t))\frac{R\left(\tau_1^{-1}(\sigma(t)),t_1\right)}{R(t,t_1)} - \frac{1}{\eta(t)r(t)}w^2(t) \quad (3.12)$$

for $t \ge t_4$. The remainder of the proof is similar to the first part of the proof of Theorem 2.2, and so we omit the details. □

The next two theorems are counterparts to Theorems 2.3 and 2.4 in the previous section.

Theorem 3.2 *Let conditions* (i), (ii), *and* (iii_b) *hold and* $\sigma(t) \le \tau_1(t)$. *If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_T^t\left[\eta(s)q(s)\phi(\sigma(s))\frac{R\left(\tau_1^{-1}(\sigma(s)),t_1\right)}{R(s,t_1)} - \frac{\eta'_+(s)}{R(s,t_1)}\right]ds = \infty \quad (3.13)$$

for all sufficiently large $t_1 \in [t_0,\infty)$ *and for all* $T > t_1$, *then Eq.* (1.1) *is oscillatory.*

Proof The proof follows from (2.2), (2.5), (2.12), (3.12), the fact that $w(t) > 0$, and Theorem 3.1, and so we omit the details. □

Theorem 3.3 *Assume that conditions* (i), (ii), *and* (iii_b) *hold and* $\sigma(t) \le \tau_1(t)$, *and let* $h, H : D \to \mathbb{R}$ *be continuous functions such that* H *belongs to the class* $\mathcal{P}$ *and* (2.22) *holds. If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[N(t,s) - \frac{\eta(s)r(s)\Psi^2(t,s)}{4}\right]ds = \infty \quad (3.14)$$

for all sufficiently large $t_1 \in [t_0,\infty)$ *and all* $T > t_1$, *where*

$$N(t,s) = H(t,s)\eta(s)q(s)\phi(\sigma(s))\frac{R\left(\tau_1^{-1}(\sigma(s)),t_1\right)}{R(s,t_1)},$$

and $\Psi(t,s)$ *is given in* (2.23)*, then Eq.* (1.1) *is oscillatory.*

Proof The proof follows from (3.12) and Theorem 2.4, and so we omit the details. □

From Theorem 3.3, we immediately have the following oscillation criterion.

Corollary 3.4 *Suppose that all conditions of Theorem 3.3 are satisfied with* (3.14) *replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t N(t,s)ds = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \eta(s)r(s)\Psi^2(t,s)ds < \infty.$$

Then Eq. (1.1) *is oscillatory.*

In the results that follow, we will be asking that $\sigma \geq \tau_1$.

Theorem 3.5 *Let conditions* (i)*,* (ii)*, and* (iii_b) *hold and* $\sigma(t) \geq \tau_1(t)$*. If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left[\eta(s)q(s)\phi(\sigma(s)) - \frac{\eta(s)r(s)\xi^2(s)}{4}\right] ds = \infty \tag{3.15}$$

for all sufficiently large $T \in (t_0,\infty)$*, where* ξ *is given in* (2.5)*, then Eq.* (1.1) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of (1.1) with $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. As in the proof of Theorem 2.2, we have the cases: (I) $z'(t) > 0$ for $t \geq t_2$ or (II) $z'(t) < 0$ for $t \geq t_2$. If case (II) holds, an argument like the one in the proof of Theorem 2.2 gives a contradiction to the positivity of z.

If case (I) holds, we proceed as in the proof of Theorem 3.1 to arrive at (3.10) for $t \geq t_4$. Using the fact that τ_1 is strictly increasing and taking $\sigma(t) \geq \tau_1(t)$ into account, we see that

$$\tau_1^{-1}(\sigma(t)) \geq t.$$

From this and the fact that z is increasing, we obtain

$$\frac{z\left(\tau_1^{-1}(\sigma(t))\right)}{z(t)} \geq 1. \tag{3.16}$$

Using (3.16) in (3.10) yields

$$w'(t) \leq \xi(t)w(t) - \eta(t)q(t)\phi(\sigma(t)) - \frac{1}{\eta(t)r(t)}w^2(t). \tag{3.17}$$

The remainder of the proof is similar to the first part of the proof of Theorem 2.2. □

Theorem 3.6 *Let conditions* (i), (ii), *and* (iii_b) *hold and* $\sigma(t) \geq \tau_1(t)$. *If there exists a positive function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left[\eta(s)q(s)\phi(\sigma(s)) - \frac{\eta'_+(s)}{R(s,t_1)}\right] ds = \infty \tag{3.18}$$

for all sufficiently large $t_1 \in [t_0, \infty)$ *and for all* $T > t_1$, *then Eq.* (1.1) *is oscillatory.*

Proof The proof follows from (2.2), (2.5), (2.12), (3.17), the fact that $w(t) > 0$, and Theorem 3.2. □

Theorem 3.7 *Assume that conditions* (i), (ii), *and* (iii_b) *hold,* $\sigma(t) \geq \tau_1(t)$, *and let* $h, H : D \to \mathbb{R}$ *be continuous functions such that* H *belongs to the class* $\mathcal{P}$ *and* (2.22) *holds. If there exists a positive function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\eta(s)q(s)\phi(\sigma(s)) - \frac{\eta(s)r(s)\Psi^2(t,s)}{4}\right] ds = \infty, \tag{3.19}$$

for all sufficiently large $T \in (t_0, \infty)$, *where* $\Psi(t,s)$ *is given in* (2.23), *then every solution of* (1.1) *is oscillatory.*

Proof The proof follows from (3.17) and Theorem 2.4, and so we omit the details. □

Corollary 3.8 *Suppose that all conditions of Theorem* 3.7 *are satisfied with* (3.19) *replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\eta(s)q(s)\phi(\sigma(s))ds = \infty \tag{3.20}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \eta(s)r(s)\Psi^2(t,s)ds < \infty. \tag{3.21}$$

Then Eq. (1.1) *is oscillatory.*

We conclude this paper with two examples to illustrate our results. The first example is concerned with the case where $p_i(t) \to \infty$ as $t \to \infty$ for $i = 1, 2$. In the second example we have $p_1(t) \to \infty$ as $t \to \infty$ and $p_2(t)$ is a constant function.

Example 3.9 Consider the second order damped neutral differential equation

$$z''(t) + \frac{1}{t}z'(t) + \left(t^3 + t\right) x\left(\frac{t}{3}\right) = 0, \quad t \geq 2, \tag{3.22}$$

with

$$z(t) = x(t) + tx\left(\frac{t}{2}\right) + 8tx(2t).$$

Here we have $r(t) = 1$, $p(t) = 1/t$, $p_1(t) = t$, $p_2(t) = 8t$, $\tau_1(t) = t/2$, $\tau_2(t) = 2t$, $\sigma(t) = t/3$, and $q(t) = t^3 + t$. It is clear that conditions $(i) - (iii_a)$, and (1.2) hold,

$$\tau_2^{-1}(t) = t/2,\ \tau_2^{-1}(\tau_2^{-1}(t)) = t/4,\ \tau_2^{-1}(\tau_1(\tau_2^{-1}(t))) = t/8,\ R(t, t_1) = t - 2,$$

and

$$\psi(t) = \frac{1}{4t}\left[1 - \frac{1}{2t} - \frac{t/2}{t}\right] = \frac{t-1}{8t^2} > 0 \quad \text{for } t \geq 2.$$

Letting $\eta(t) = c > 0$ be a constant, and choosing $t_1 = t_0 = 2$ and $T \in (2, \infty)$, it follows that

$$\begin{aligned}\limsup_{t\to\infty} \int_T^t \left[\eta(s)q(s)\psi(\sigma(s))\frac{R\left(\tau_2^{-1}(\sigma(s)), t_1\right)}{R(s, t_1)} - \frac{\eta(s)r(s)\xi^2(s)}{4}\right] ds \\ = \limsup_{t\to\infty} \int_T^t \left[c(s^3 + s)\left(\frac{3s-9}{8s^2}\right)\frac{s-12}{6s-12} - \frac{c}{4s^2}\right] ds = \infty,\end{aligned}$$

since

$$\limsup_{t\to\infty} \int_T^t c(s^3 + s)\left(\frac{3s-9}{8s^2}\right)\frac{s-12}{6s-12} ds = \infty,$$

and

$$\limsup_{t\to\infty} \int_T^t \frac{c}{4s^2} ds = \frac{c}{4T} < \infty,$$

i.e., (2.4) holds. Therefore, by Theorem 2.2, Eq. (3.22) is oscillatory.

Example 3.10 Consider the equation

$$\left(\frac{1}{t}z'(t)\right)' + \frac{1}{t^2}z'(t) + t^4 x(2t) = 0, \quad t \geq 2, \tag{3.23}$$

with

$$z(t) = x(t) + t^2 x\left(\frac{t}{2}\right) + x(3t).$$

Here $r(t) = 1/t$, $p(t) = 1/t^2$, $p_1(t) = t^2$, $p_2(t) = 1$, $\tau_1(t) = t/2$, $\tau_2(t) = 3t$, $\sigma(t) = 2t$, and $q(t) = t^4$. It is clear that conditions (i), (ii), (iii_b), and (1.2) hold,

$$\tau_1^{-1}(t) = 2t, \tau_1^{-1}(\tau_1^{-1}(t)) = 4t, \tau_1^{-1}(\tau_2(\tau_1^{-1}(t))) = 12t, R(t, t_1) = (t^2 - 4)/2,$$

and

$$\phi(t) = \frac{1}{4t^2}\left[1 - \frac{1}{16t^2}\frac{16t^2 - 4}{4t^2 - 4} - \frac{1}{144t^2}\frac{144t^2 - 4}{4t^2 - 4}\right] = \frac{72t^4 - 108t^2 + 5}{288t^4(t^2 - 1)} > 0,$$

for $t \geq 2$. Letting $H(t, s) = (t - s)^2$, we see that $H \in \mathcal{P}$ and $h(t, s) = 2$. With $\eta(t) = 1$, we have $\Psi(t, s) = -1 - t/s$. Then, with $t_1 = t_0 = 2$, for all $T \in (2, \infty)$, we see that

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_T^t H(t, s)\eta(s)q(s)\phi(\sigma(s))ds \\ &= \limsup_{t\to\infty} \frac{1}{(t - T)^2} \int_T^t (t - s)^2 \frac{1152s^4 - 432s^2 + 5}{4608(4s^2 - 1)} ds = \infty\end{aligned}$$

and

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_T^t \eta(s)r(s)\Psi^2(t, s)ds \\ &= \limsup_{t\to\infty} \frac{1}{(t - T)^2} \int_T^t \frac{1}{s}\left(1 + \frac{t}{s}\right)^2 ds = \frac{1}{2T^2} < \infty,\end{aligned}$$

i.e., (3.20) and (3.21) hold. Thus, Eq. (3.23) is oscillatory by Corollary 3.8.

References

1. Agarwal, R.P., Grace, S.R., O'Regan, D.: The oscillation of certain higher-order functional differential equations. Math. Comput. Model. **37**, 705–728 (2003)
2. Agarwal, R.P., Bohner, M., Li, T., Zhang, C.: Oscillation of second-order Emden–Fowler neutral delay differential equations. Ann. Math. Pura Appl. **193**, 1861–1875 (2014)
3. Agwa, H.A., Khodier, A.M.M., Arafa, H.M.: Oscillation of second-order nonlinear neutral dynamic equations with mixed arguments on time scales. J. Basic Appl. Res. Int. **17**, 49–66 (2016)
4. Arul, R., Shobha, V.S.: Oscillation of second order nonlinear neutral differential equations with mixed neutral term. J. Appl. Math. Phys. **3**, 1080–1089 (2015)
5. Baculíková, B., J. Džurina, : Oscillation theorems for second-order neutral differential equations. Comput. Math. Appl. **61**, 94–99 (2011)
6. Bohner, M., Grace, S.R., Jadlovská, I.: Oscillation criteria for second-order neutral delay differential equations. Electron. J. Qual. Theory. Differ. Equ. **2017**, 1–12 (2017)
7. Chen, D.-X.: Oscillation of second-order Emden–Fowler neutral delay dynamic equations on time scales. Math. Comput. Model. **51**, 1221–1229 (2010)
8. Grace, S.R., Graef, J.R., Tunç, E.: Oscillatory behavior of second order damped neutral differential equations with distributed deviating arguments. Miskolc Math. Notes **18**, 759–769 (2017)
9. Ji, T., Tang, S., Thandapani, E.: Oscillation of second-order neutral dynamic equations with mixed arguments. Appl. Math. Inf. Sci. **8**, 2225–2228 (2014)

10. Kubiaczyk, I., Saker, S.H., Sikorska-Nowak, A.: Oscillation criteria for nonlinear neutral functional dynamic equations on time scales. Math. Slovaca **63**, 263–290 (2013)
11. Li, T., Agarwal, R.P., Bohner, M.: Some oscillation results for second-order neutral dynamic equations. Hacet. J. Math. Stat. **41**, 715–721 (2012)
12. Li, T., Rogovchenko, Y.V.: Oscillatory behavior of second-order nonlinear neutral differential equations. Abstr. Appl. Anal., **2014** (2014), Article ID: 143614, 1–8
13. Li, T., Rogovchenko, Y.V.: Oscillation of second-order neutral differential equations. Math. Nachr. **288**, 1150–1162 (2015)
14. Li, T., Rogovchenko, Y.V.: Oscillation criteria for second-order superlinear Emden–Fowler neutral differential equations. Monatsh. Math. **184**, 489–500 (2017)
15. Li, T., Thandapani, E., Graef, J.R., Tunç, E.: Oscillation of second-order Emden–Fowler neutral differential equations. Nonlinear Stud. **20**, 1–8 (2013)
16. Li, T., Senel, M.T., Zhang, C.: Oscillation of solutions to second-order half-linear differential equations with neutral terms. Electron. J. Differ. Equ. **2013**, 1–7 (2013)
17. Li, T.: Comparison theorems for second-order neutral differential equations of mixed type. Electron. J. Differ. Equ. **2010**, 1–7 (2010)
18. Li, T., Baculíková, B., Džurina, J.: Oscillation results for second-order neutral differential equations of mixed type. Tatra Mt. Math. Publ. **48**, 101–116 (2011)
19. Qi, Y., Yu, J.: Oscillation of second order nonlinear mixed neutral differential equations with distributed deviating arguments. Bull. Malays. Math. Sci. Soc. **38**, 543–560 (2015)
20. Philos, ChG: Oscillation theorems for linear differential equations of second order. Arch. Math. **53**, 482–492 (1989)
21. Thandapani, E., Rama, R.: Comparison and oscillation theorems for second order nonlinear neutral differential equations of mixed type. Serdica Math. J. **39**, 1–16 (2013)
22. Thandapani, E., Selvarangam, S., Vijaya, M., Rama, R.: Oscillation results for second order nonlinear differential equation with delay and advanced arguments. Kyungpook Math. J. **56**, 137–146 (2016)
23. Wang, R., Li, Q.: Oscillation and asymptotic properties of a class of second-order Emden–Fowler neutral differential equations. Springer Plus **2016**, 1–15 (2016)
24. Zhang, C., Baculíková, B., Džurina, J., Li, T.: Oscillation results for second-order mixed neutral differential equations with distributed deviating arguments. Math. Slovaca **66**, 1–12 (2016)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.