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# Discontinuous Sturm–Liouville Problems with Eigenparameter-Dependent Boundary and Transmissions Conditions ⋆

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**Abstract.** The purpose of this paper is to study a Sturm–Liouville problem with discontinuities in the case when an eigenparameter appears not only in the differential equation but it also appears in both the boundary and transmission conditions.

We suggest a new approach for the definition of a suitable Hilbert space and a symmetric linear operator defined in this space in such a way that the considered problem can be interpreted as the eigenvalue problem of this operator and for construction and approximation of a fundamental solution. We apply these results to find asymptotic formulas of eigenvalues and corresponding eigenfunctions.

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**Key words:** discontinuous Sturm–Liouville problems, eigenvalues, eigenfunctions, transmission conditions, eigenparameter-dependent boundary conditions.

## 1. Introduction

It is well-known that many topics in mathematical physics require the investigation of the eigenvalues and eigenfunctions of Sturm–Liouville type boundary-value problems. Therefore, the Sturmian theory is one of the most actual and extensively developing fields of theoretical and applied mathematics. Particularly, in recent years, highly important results in this field have been obtained for the case when the eigenparameter appears not only in the differential equation but also in the boundary conditions. The literature on such results is voluminous and we refer to [1, 2, 11, 17] and corresponding bibliography cited therein. In particular, [2, 4, 11, 12, 14] and [15] contains many references to problems in physics and mechanics. It should be mentioned that in the series of S. and Y. Yakubov's works published in recent years they have constructed an abstract theory of boundary-value problems with an eigenparameter in the boundary conditions (see [16–18] and

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corresponding bibliography). While the general theory and methods of boundary-value problems with continuous coefficients are highly developed, very little is known about a general character of similar problems with discontinuities.

The main aim of this study is to extend some results of the standard Sturm–Liouville problems to the discontinuous case.

Namely, we shall investigate the Sturm–Liouville equation

$$\tau u := -p(x)u'' + q(x)u = \lambda u, \tag{1}$$

to hold in finite interval $(a, b)$ except one inner point $c \in (a, b)$, subject to the eigenparameter-dependent boundary conditions

$$L_1(u) := \lambda(\alpha_1' u(a) - \alpha_2' u'(a)) - (\alpha_1 u(a) - \alpha_2 u'(a)) = 0, \tag{2}$$

$$L_2(u) := \lambda(\beta_1' u(b) - \beta_2' u'(b)) + (\beta_1 u(b) - \beta_2 u'(b)) = 0 \tag{3}$$

and the eigenparameter-dependent transmission conditions

$$L_3(u) := u(c+0) - u(c-0) = 0, \tag{4}$$

$$L_4(u) := \gamma_2 u'(c+0) - \gamma_1 u'(c-0) + (\lambda\delta_1 + \delta_2)u(c) = 0, \tag{5}$$

where $p(x) = 1/p_1^2$ for $x \in [a, c)$ and $p(x) = 1/p_2^2$ for $x \in (c, b]$; $\lambda$ is complex eigenparameter; $q(x)$ is real-valued and continuous in both $[a, c)$ and $(c, b]$, and has finite limits $q(c \pm 0) := \lim_{x \to c \pm 0} q(x)$; $p_i, \alpha_i, \alpha_i', \beta_i, \beta_i'$ and $\delta_i$ $(i = 1, 2)$ are real numbers.

Consequently, we are interested in two types of generalizations of classical Sturm–Liouville problems:

First, we wish to consider more general differential equations, for which the coefficient at the highest derivative may have the discontinuity at one inner point of the considered interval. Naturally, in this point of discontinuity two supplementary transmission conditions are given.

A second generalization allows boundary and transmission conditions depending on an eigenparameter.

It must be noted that some special cases of the considered problem (1)–(5) arise after an application of the method of separation of variables to the varied assortment of physical problems. For example, some boundary-value problems with transmission conditions arise in heat and mass transfer problems (see, for example, [4]), in vibrating string problems when the string loaded additionally with point masses (see, for example, [12]) and in diffraction problems (see,for example, [15]). Properties such as isomorphism, coerciveness with respect to the spectral parameter, completeness and Abel bases of a system of root functions, distributions of eigenvalues of some discontinuous boundary value problems with transmission conditions and its applications to the corresponding initial-boundary-value problems for parabolic equations have been investigated in [5–8, 14] and [17]. Also, some problems with transmission conditions which arise in mechanics (thermal conduction problem for a thin laminated plate) were studied in the article [14].

Similar problems for differential equations with discontinuous coefficients were investigated by Rasulov in monographs [9, 10]. But, the considered discontinuous problems in these works do not contain transmission conditions.

## 2. An Operator Formulation in the Adequate Hilbert Space

In this section we shall introduce the special inner product in the Hilbert space $H := L_2[a, b] \oplus C^3$ and a symmetric linear operator $A$ defined on this Hilbert space such a way that the problem (1)–(5) can be considered as the eigenvalue problem of this operator. Throughout this paper we shall assume that the coefficients $\gamma_1$, $\gamma_2$ and $\delta_1$ have the same sign (without losing the generality we shall assume that $\gamma_1$, $\gamma_2$ and $\delta_1$ are positive),

$$\rho_1 := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} > 0 \quad \text{and} \quad \rho_2 := \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0.$$

Let us introduce a new equivalent inner product on $H = L_2[a, b] \oplus C^3$ by

$$\begin{aligned} \langle F, G \rangle \;:=\; & p_1^2 \gamma_1 \int_a^c f(x)\overline{g(x)}\,\mathrm{d}x \;+ \\ & + p_2^2 \gamma_2 \int_c^b f(x)\overline{g(x)}\,\mathrm{d}x + \frac{\gamma_1}{\rho_1} f_1 \overline{g_1} + \frac{\gamma_2}{\rho_2} f_2 \overline{g_2} + \frac{1}{\delta_1} f_3 \overline{g_3} \end{aligned}$$

for

$$F := (f(x), f_1, f_2, f_3), \qquad G := (g(x), g_1, g_2, g_3) \in H$$

which is connected with coefficients of our problem. For a short exposition we shall use the following notations:

$$\begin{aligned} B_a(u) \;&:=\; \alpha_1 u(a) - \alpha_2 u'(a),\; B_a'(u) := \alpha_1' u(a) - \alpha_2' u'(a), \\ B_b(u) \;&:=\; \beta_1 u(b) - \beta_2 u'(b),\; B_b'(u) := \beta_1' u(b) - \beta_2' u'(b), \\ T_c(u) \;&:=\; \gamma_2 u'(c+0) - \gamma_1 u'(c-0) + \delta_2 u(c), \text{ and } T_c'(u) := -\delta_1 u(c). \end{aligned}$$

In the Hilbert space $H$ we define a linear operator $A$ with domain of definition

$$\begin{aligned} D(A) := \Big\{ & F = (f(x), f_1, f_2, f_3) : f \text{ is absolutely continuous in } [a, b],\ f' \\ & \text{is absolutely continuous on both } [a, c) \text{ and } (c, b], \text{ and has a finite} \\ & \text{limits } f'(c \pm 0) = \lim_{x \to c \pm 0} f'(x),\ \tau f \in L_2[a, b],\ f_1 = B_a'(f),\ f_2 = \\ & B_b'(f),\ f_3 = T_c'(f) \Big\} \end{aligned}$$

and action law

$$AF = (\tau f, B_a(f), -B_b(f), T_c(f)).$$

Thus, the problem (1)–(5) can be written in the form

$$AF = \lambda F,$$

where $F = (f(x), B'_a(f), B'_b(f), T'_c(f)) \in D(A)$. Naturally, by eigenvalues and eigenfunctions of the problem (1)–(5) we mean eigenvalues and first components of corresponding eigenelements of the operator $A$, respectively.

THEOREM 1. *The linear operator A is symmetric.*

*Proof.* Let $F, G \in D(A)$. By two partial integration we obtain

$$\begin{aligned}\langle AF, G\rangle \;=\; & \langle F, AG\rangle + \gamma_1 W(f, \overline{g}; c) - \gamma_1 W(f, \overline{g}; a) + \gamma_2 W(f, \overline{g}; b) \;- \\ & - \gamma_2 W(f, \overline{g}; c) + \frac{\gamma_1}{\rho_1}(B_a(f)B'_a(\overline{g}) - B'_a(f)B_a(\overline{g})) \;+ \\ & + \frac{\gamma_2}{\rho_2}(B'_b(f)B_b(\overline{g}) - B_b(f)B'_b(\overline{g})) + \frac{1}{\delta_1}(T_c(f)T'_c(\overline{g}) \;- \\ & - T'_c(f)T_c(\overline{g})), \end{aligned} \tag{6}$$

where, as usual, $W(f, g; x)$ is denoted the Wronskians $f(x)g'(x) - f'(x)g(x)$.

It is easy to show that

$$B_a(f)B'_a(\overline{g}) - B'_a(f)B_a(\overline{g}) = \rho_1 W(f, \overline{g}; a),$$
$$B'_b(f)B_b(\overline{g}) - B_b(f)B'_b(\overline{g}) = -\rho_2 W(f, \overline{g}; b)$$

and

$$T_c(f)T'_c(\overline{g}) - T'_c(f)T_c(\overline{g}) = -\delta_1\gamma_1 W(f, \overline{g}; c) + \delta_1\gamma_2 W(f, \overline{g}; c).$$

Substituting into (6) we have

$$\langle AF, G\rangle = \langle F, AG\rangle \quad (F, G \in D(A)), \tag{7}$$

so $A$ is symmetric. □

COROLLARY 2. *All eigenvalues of the problem* (1)–(5) *are real.*

We can now assume that all eigenfunctions of the problem (1)–(5) are real-valued.

COROLLARY 3. *Let $\lambda_1$ and $\lambda_2$ be two different eigenvalues of the problem* (1)–(5). *Then the corresponding eigenfunctions $u_1$ and $u_2$ of this problem are orthogonal in the sense of*

$$\begin{aligned} & p_1^2\gamma_1 \int_a^c u_1(x)u_2(x)\,\mathrm{d}x + p_2^2\gamma_2 \int_c^b u_1(x)u_2(x)\,\mathrm{d}x + \frac{\gamma_1}{\rho_1}B'_a(u_1)B'_a(u_2) + \\ & \quad + \frac{\gamma_2}{\rho_2}B'_b(u_1)B'_b(u_2) + \frac{1}{\delta_1}T'_c(u_1)T'_c(u_2) = 0. \end{aligned}$$

## 3. Construction and Asymptotic Approximation of Fundamental Solutions

For next consideration we need to following lemma which can be proved by modifying of the method of proving of the Theorem 1.5 in [13].

LEMMA 4. *Let the real valued function $q(x)$ be continuous in $[a, b]$ and $f(\lambda)$, $g(\lambda)$ are given entire functions. Then for any $\lambda \in C$ the equation*

$$\tau u := -p(x)u'' + q(x)u = \lambda u, \quad x \in [a, b]$$

*has a unique solution $u = u(x, \lambda)$ satisfying the initial conditions*

$$u(a) = f(\lambda), u'(a) = g(\lambda) \quad (\text{or } u(b) = f(\lambda), u'(b) = g(\lambda)).$$

*For each fixed $x \in [a, b]$, $u(x, \lambda)$ is an entire function of $\lambda$.*

Let $\phi_{1\lambda}(x) := \phi_1(x, \lambda)$ be the solution of Equation (1) in the interval $[a, c]$ satisfying the initial conditions

$$u(a) = -\alpha_2 + \lambda\alpha_2', \qquad u'(a) = -\alpha_1 + \lambda\alpha_1'. \tag{8}$$

Now we can define the solution $\phi_{2\lambda}(x) := \phi_2(x, \lambda)$ of Equation (1) on the interval $[c, b]$ in terms of $\phi_1(c - 0, \lambda)$ and $\phi_1'(c - 0, \lambda)$ by the initial conditions

$$\begin{aligned} &u(c) = \phi_1(c - 0, \lambda), \\ &u'(c) = \frac{\gamma_1}{\gamma_2}\phi_1'(c - 0, \lambda) - \left(\frac{\lambda\delta_1 + \delta_2}{\gamma_2}\right)\phi_1(c - 0, \lambda). \end{aligned} \tag{9}$$

Analogically, we can define the solutions $\chi_{1\lambda}(x)$ and $\chi_{2\lambda}(x)$ by initial conditions

$$\chi_{2\lambda}(b) = \beta_2 + \lambda\beta_2', \qquad \chi_{2\lambda}'(b) = \beta_1 + \lambda\beta_1' \tag{10}$$

and

$$\begin{aligned} &\chi_{1\lambda}(c) = \chi_2(c + 0, \lambda), \\ &\chi_{1\lambda}'(c) = \frac{\gamma_2}{\gamma_1}\chi_2'(c + 0, \lambda) + \left(\frac{\lambda\delta_1 + \delta_2}{\gamma_1}\right)\chi_2(c + 0, \lambda), \end{aligned} \tag{11}$$

respectively.

Finally, let us define two "fundamental" solutions of Equation (1) on whole $[a, c) \cup (c, b]$ as

$$\begin{aligned} \phi(x, \lambda) &= \begin{cases} \phi_1(x, \lambda) & \text{for } x \in [a, c), \\ \phi_2(x, \lambda) & \text{for } x \in (c, b], \end{cases} \\ \chi(x, \lambda) &= \begin{cases} \chi_1(x, \lambda) & \text{for } x \in [a, c), \\ \chi_2(x, \lambda) & \text{for } x \in (c, b]. \end{cases} \end{aligned} \tag{12}$$

It must note that each of this solutions satisfy both transmission conditions (4) and (5). Moreover, $\phi(x, \lambda)$ satisfies one of the boundary conditions (namely the condition (2)), but $\chi(x, \lambda)$ the other boundary condition (3).

Let us consider the Wronskians

$$\begin{aligned} w_i(\lambda) &:= W_\lambda(\phi_i, \chi_i; x) \\ &:= \phi_i(x,\lambda)\chi_i'(x,\lambda) - \phi_i'(x,\lambda)\chi_i(x,\lambda) \quad (i = 1, 2) \end{aligned}$$

which are independent of $x$ and are entire functions. The short calculation gives

$$\gamma_1 w_1(\lambda) = \gamma_2 w_2(\lambda).$$

Now we may introduce to the consideration the characteristic function $w(\lambda)$ as

$$w(\lambda) := \gamma_1 w_1(\lambda) = \gamma_2 w_2(\lambda).$$

THEOREM 5. *The eigenvalues of the problem* (1)–(5) *are consist of the zeros of the functions*

$$w(\lambda) \text{ and } \Delta(\lambda) := \frac{\gamma_2}{\gamma_1} w(\lambda) + (\lambda\delta_1 + \delta_2)\chi_{2\lambda}(c)\phi_{2\lambda}(c).$$

*Proof.* Let $\lambda_0$ be an eigenvalue and $u_0(x)$ be any corresponding eigenfunction. Show that $w(\lambda_0) = 0$ or $\Delta(\lambda_0) = 0$. Let us assume the contrary, that $w(\lambda_0) \neq 0$ and $\Delta(\lambda_0) \neq 0$. Then this implies that $W(\phi_i(x, \lambda_0), \chi_i(x, \lambda_0)) \neq 0$ $(i = 1, 2)$. Consequently, each pair of functions $\phi_1(x, \lambda_0)$, $\chi_1(x, \lambda_0)$ and $\phi_2(x, \lambda_0)$, $\chi_2(x, \lambda_0)$ are linearly independent. Therefore eigenfunction $u_0(x)$ can be written in the form

$$u_0(x) = \begin{cases} c_1\phi_1(x, \lambda_0) + c_2\chi_1(x, \lambda_0) & \text{for } x \in [a, c), \\ c_3\phi_2(x, \lambda_0) + c_4\chi_2(x, \lambda_0) & \text{for } x \in (c, b], \end{cases}$$

where at least one of the constants $c_1, c_2, c_3, c_4$ is not zero. By substituting this representation in the conditions (2)–(5) we obtain a system of linear, homogenous equations for the determination of the constants $c_1, c_2, c_3, c_4$. By routine calculation we see that the determinant of this system is equal to $w(\lambda)\Delta(\lambda)$, which is not zero by assumption. Hence this system of equations has only trivial solution $c_1 = c_2 = c_3 = c_4 = 0$, and so we reach a contradiction. Thus it is shown that each eigenvalue is zero of the functions $w(\lambda)$ or $\Delta(\lambda)$. Now we must show that if $w(\lambda_0)\Delta(\lambda_0) = 0$, then $\lambda_0$ is an eigenvalue. Consider the two possible cases $w(\lambda_0) = 0$ and $w(\lambda_0) \neq 0$, $\Delta(\lambda_0) = 0$ separately. If $w(\lambda_0) = 0$ then $W(\phi_1(x, \lambda_0), \chi_1(x, \lambda_0)) = w_1(\lambda_0) = 0$, and consequently the functions $\phi_1(x, \lambda_0)$, $\chi_1(x, \lambda_0)$ are linearly dependent solutions of Equation (1) in the interval $[a, c)$, i.e.

$$\chi_1(x, \lambda_0) = k_1\phi_1(x, \lambda_0)$$

for some $k_1 \neq 0$. In this case, since $\phi_1(x, \lambda_0)$ satisfies the boundary condition (2), $\chi_1(x, \lambda_0)$ also satisfies boundary condition (2). Thus, the solution $\chi(x, \lambda_0)$ of Equation (1) defined in (12) satisfies all boundary and transmission conditions (2)–(5). Thus $\chi(x, \lambda_0)$ would be any eigenfunction for eigenvalue $\lambda_0$.

Finally if $w(\lambda_0) \neq 0$ and $\Delta(\lambda_0) = 0$, then, since $W(\phi_i(x, \lambda_0), \chi_i(x, \lambda_0)) = w_i(\lambda_0) \neq 0$, each of pair functions $\phi_1(x, \lambda_0)$, $\chi_1(x, \lambda_0)$ and $\phi_2(x, \lambda_0)$, $\chi_2(x, \lambda_0)$ are linearly independent. As a result, the general solution of Equation (1) may be expressed as

$$u(x, \lambda_0) = \begin{cases} c_1\phi_1(x, \lambda_0) + c_2\chi_1(x, \lambda_0) & \text{for } x \in [a, c), \\ c_3\phi_2(x, \lambda_0) + c_4\chi_2(x, \lambda_0) & \text{for } x \in (c, b], \end{cases} \tag{13}$$

where $c_1, c_2, c_3, c_4$ are arbitrary constants. Again, substituting (13) in the conditions (2)–(4) we obtain a system of linear, homogenous equations for determination of the constants $c_1, c_2, c_3, c_4$, the determinant of which is equal to $w(\lambda_0)\Delta(\lambda_0)$, which is equal to zero by assumption. Hence there is nontrivial solution $(c_1, c_2, c_3, c_4) \neq (0, 0, 0, 0)$ of these system of equations. So the corresponding solution $u(x, \lambda_0)$ would be an eigenfunction for the eigenvalue $\lambda_0$. The proof is completed. □

LEMMA 6. *Let $\lambda = s^2$. Then the following integral and integro-differential equations hold for $k = 0$ and $k = 1$:*

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\phi_{1\lambda}(x) &= (-\alpha_2 + s^2\alpha_2')\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos[p_1 s(x-a)] - \\ &\quad - \frac{1}{p_1 s}(-\alpha_1 + s^2\alpha_1')\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin[p_1 s(x-a)] + \\ &\quad + \frac{p_1}{s}\int_a^x \frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin[p_1 s(x-y)]q(y)\phi_{1\lambda}(y)\,\mathrm{d}y, \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\phi_{2\lambda}(x) &= \phi_{1\lambda}(c)\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos[p_2 s(x-c)] + \\ &\quad + \frac{1}{p_2 s}\left[\frac{\gamma_1}{\gamma_2}\phi_{1\lambda}'(c) - \frac{(s^2\delta_1 + \delta_2)}{\gamma_2}\phi_{1\lambda}(c)\right]\frac{\mathrm{d}^k}{\mathrm{d}x^k} \times \\ &\quad \times \sin[p_2 s(x-c)] + \\ &\quad + \frac{p_2}{s}\int_c^x \frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin[p_2 s(x-y)]q(y)\phi_{2\lambda}(y)\,\mathrm{d}y. \end{aligned} \tag{15}$$

*Proof.* For proving it is enough substitute $s^2\phi_{1\lambda}(y) + p(y)\phi_{1\lambda}''(y)$ and $s^2\phi_{2\lambda}(y) + p(y)\phi_{2\lambda}''(y)$ instead of $q(y)\phi_{1\lambda}(y)$ and $q(y)\phi_{2\lambda}(y)$ in the integral terms of the (14) and (15) respectively and integrate by parts twice. □

LEMMA 7. *Let $\lambda = s^2$, $\operatorname{Im} s = t$. Then if $\alpha_2' \neq 0$*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\phi_{1\lambda}(x) = s^2\alpha_2'\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos[p_1 s(x-a)] + \mathrm{O}\big(|s|^{k+1}\mathrm{e}^{|t|p_1(x-a)}\big), \tag{16}$$

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\phi_{2\lambda}(x) &= -s^3\alpha_2'\frac{\delta_1}{p_2\gamma_2}\cos[p_1 s(c-a)]\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin[p_2 s(x-c)] + \\ &\quad + \mathrm{O}\big(|s|^{k+2}\mathrm{e}^{|t|(p_2(x-c)+p_1(c-a))}\big) \end{aligned} \tag{17}$$

*as* $|\lambda| \to \infty$, *while if* $\alpha_2' = 0$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\phi_{1\lambda}(x) = -\frac{s\alpha_1'}{p_1}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin[p_1 s(x-a)] + \mathrm{O}\big(|s|^k \mathrm{e}^{|t|p_1(x-a)}\big), \tag{18}$$

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\phi_{2\lambda}(x) &= \frac{s^2\alpha_1'\delta_1}{p_1 p_2 \gamma_2}\sin[p_1 s(c-a)]\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin[p_2 s(x-c)] + \\ &\quad + \mathrm{O}\big(|s|^{k+1}\mathrm{e}^{|t|(p_2(x-c)+p_1(c-a))}\big)\end{aligned} \tag{19}$$

*as* $|\lambda| \to \infty$ $(k = 0, 1)$. *Each of these asymptotic equalities holds uniformly for* $x$.

*Proof.* The asymptotic formulas for $\phi_{1\lambda}(x)$ can be found easily by applying the Titchmarsh's lemma [13, Lemma 1.7]. But the proof of the formulas (17) and (19) need special consideration since the function $\phi_{2\lambda}(x)$ is defined by the special type initial conditions of the form (9).

Let $\alpha_2' \neq 0$. Substituting (16) in (15) (for $k = 0$) and taking into account that

$$|\sin z| = \mathrm{O}\big(\mathrm{e}^{|\mathrm{Im}\, z|}\big) \quad \text{and} \quad |\cos z| = \mathrm{O}\big(\mathrm{e}^{|\mathrm{Im}\, z|}\big)$$

we get

$$\begin{aligned}\phi_{2\lambda}(x) &= \big[s^2\alpha_2'\cos[p_1 s(c-a)] + \mathrm{O}\big(|s|\mathrm{e}^{|t|p_1(c-a)}\big)\big]\cos[p_2 s(x-c)] + \\ &\quad + \frac{1}{p_2 s}\Bigg[\frac{\gamma_1}{\gamma_2}\big(-p_1 s^3\alpha_2'\sin[p_1 s(c-a)] + \mathrm{O}\big(|s|^2\mathrm{e}^{|t|p_1(c-a)}\big)\big) - \\ &\quad - \frac{(s^2\delta_1+\delta_2)}{\gamma_2}\big(s^2\alpha_2'\cos[p_1 s(c-a)] + \mathrm{O}\big(|s|\mathrm{e}^{|t|p_1(c-a)}\big)\big)\Bigg] \times \\ &\quad \times \sin[p_2 s(x-c)] + \frac{p_2}{s}\int_c^x \sin[p_2 s(x-y)]q(y)\phi_{2\lambda}(y)\,\mathrm{d}y \\ &= -s^3\frac{1}{p_2}\alpha_2'\frac{\delta_1}{\gamma_2}\cos[p_1 s(c-a)]\sin[p_2 s(x-c)] + \\ &\quad + s^2\alpha_2'(\cos[p_1 s(c-a)]\cos[p_2 s(x-c)] - \\ &\quad - \frac{p_1}{p_2}\frac{\gamma_1}{\gamma_2}\sin[p_1 s(c-a)]\sin[p_2 s(x-c)]) - \\ &\quad - s\frac{1}{p_2}\alpha_2'\frac{\delta_2}{\gamma_2}\cos[p_1 s(c-a)]\sin[p_2 s(x-c)] + \\ &\quad + \frac{p_2}{s}\int_c^x \sin[p_2 s(x-y)]q(y)\phi_{2\lambda}(y)\,\mathrm{d}y + \\ &\quad + \mathrm{O}\big(|s|^2\mathrm{e}^{|t|(p_1(c-a)+p_2(x-c))}\big).\end{aligned} \tag{20}$$

Multiplying by $|s|^{-3}\mathrm{e}^{-|t|(p_1(c-a)+p_2(x-c))}$ and denoting

$$F(x,\lambda) = |s|^{-3}\mathrm{e}^{-|t|(p_1(c-a)+p_2(x-c))}\phi_{2\lambda}(x)$$

we have the next "asymptotic integral equation"

$$F(x,\lambda) = |s|^{-3}\mathrm{e}^{-|t|(p_1(c-a)+p_2(x-c))} \times$$

$$\times\left\{-s^3\frac{1}{p_2}\alpha_2'\frac{\delta_1}{\gamma_2}\cos[p_1s(c-a)]\sin[p_2s(x-c)]+\right.$$
$$+s^2\alpha_2'\Big(\cos[p_1s(c-a)]\cos[p_2s(x-c)]-$$
$$\left.-\frac{p_1}{p_2}\frac{\gamma_1}{\gamma_2}\sin[p_1s(c-a)]\sin[p_2s(x-c)]\Big)-\right.$$
$$\left.-s\frac{1}{p_2}\alpha_2'\frac{\delta_2}{\gamma_2}\cos[p_1s(c-a)]\sin[p_2s(x-c)]\right\}+$$
$$+\frac{p_2}{s^4}\int_c^x\sin[p_2s(x-y)]q(y)\mathrm{e}^{-|t|p_2(x-y)}F(y,\lambda)\,\mathrm{d}y+\mathrm{O}\Big(\frac{1}{s}\Big).$$

Now, denoting $M(\lambda) := \max_{x\in[c,b]}|F(x,\lambda)|$, from the last equation we can derive that

$$M(\lambda)\leqslant M_0\Big(\frac{1}{p_2}\Big|\alpha_2'\frac{\delta_1}{\gamma_2}\Big|+\frac{1}{|s|}\Big)$$

for some $M_0>0$. Consequently $M(\lambda)=\mathrm{O}(1)$ as $|\lambda|\to\infty$, so

$$\phi_{2\lambda}(x)=\mathrm{O}\big(|s|^3\mathrm{e}^{|t|(p_1(c-a)+p_2(x-c))}\big)\quad\text{as }|\lambda|\to\infty. \tag{21}$$

Substituting in (20) gives (17) for case $k=0$. The case $k=1$ of the (17) follows by applying the same procedure as in the case $k=0$.

The proof of (19) is similar to that of (17) and hence omitted. □

Similarly one can establish the following lemma.

LEMMA 8. *Let* $\lambda=s^2$, $\operatorname{Im}s=t$. *Then if* $\beta_2'\neq0$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi_{1\lambda}(x) = \frac{s^3\beta_2'\delta_1}{p_1\gamma_1}\cos[p_2s(c-b)]\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin[p_1s(x-c)]+$$
$$+\mathrm{O}\big(|s|^{k+2}\mathrm{e}^{|t|(p_2(b-c)+p_1(c-x))}\big), \tag{22}$$
$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi_{2\lambda}(x) = s^2\beta_2'\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos[p_2s(x-b)]+\mathrm{O}\big(|s|^{k+1}\mathrm{e}^{|t|p_2(b-x)}\big) \tag{23}$$

*as* $|\lambda|\to\infty$, *while if* $\beta_2'=0$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi_{1\lambda}(x) = \frac{s^2\beta_1'\delta_1}{p_1p_2\gamma_1}\sin[p_2s(c-b)]\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin[p_1s(x-c)]+$$
$$+\mathrm{O}\big(|s|^{k+1}\mathrm{e}^{|t|(p_2(b-c)+p_1(c-x))}\big), \tag{24}$$
$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi_{2\lambda}(x) = \frac{s\beta_1'}{p_2}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin[p_2s(x-b)]+\mathrm{O}\big(|s|^{k+1}\mathrm{e}^{|t|p_2(b-x)}\big) \tag{25}$$

*as* $|\lambda|\to\infty$ $(k=0,1)$. *Each of these asymptotic equalities holds uniformly for* $x$.

By substituting the obtained asymptotic formulas for $\phi_i(x,\lambda)$ and $\chi_i(x,\lambda)$ $(i = 1,2)$ in the definitions of $w(\lambda)$ and $\Delta(\lambda)$ we can establish the following theorem.

THEOREM 9. *Let* $\lambda = s^2, \operatorname{Im} s = t$. *Then the characteristic functions* $w(\lambda)$ *and* $\Delta(\lambda)$ *has the following asymptotic representations:*

Case 1: *If* $\beta_2' \neq 0$ *and* $\alpha_2' \neq 0$, *then*

$$\begin{aligned}\Delta(\lambda) &= 2s^6\delta_1\alpha_1'\beta_2'\cos[p_1s(c-a)]\cos[p_2s(b-c)] + \\ &\quad + \mathrm{O}\big(|s|^5 \mathrm{e}^{|t|(p_2(b-c)+p_1(c-a))}\big), \qquad (26)\\ w(\lambda) &= s^6\frac{\delta_1\gamma_1}{\gamma_2}\alpha_1'\beta_2'\cos[p_1s(c-a)]\cos[p_2s(b-c)] + \\ &\quad + \mathrm{O}\big(|s|^5 \mathrm{e}^{|t|(p_2(b-c)+p_1(c-a))}\big). \qquad (27)\end{aligned}$$

Case 2: *If* $\beta_2' \neq 0$ *and* $\alpha_2' = 0$, *then*

$$\begin{aligned}\Delta(\lambda) &= -\frac{2}{p_1}s^5\delta_1\alpha_1'\beta_2'\sin[p_1s(c-a)]\cos[p_2s(b-c)] + \\ &\quad + \mathrm{O}\big(|s|^4 \mathrm{e}^{|t|(p_2(b-c)+p_1(c-a))}\big), \qquad (28)\\ w(\lambda) &= -s^5\alpha_1'\beta_2'\frac{\gamma_1\delta_1}{\gamma_2 p_1}\sin[p_1s(c-a)]\cos[p_2s(b-c)] + \\ &\quad + \mathrm{O}\big(|s|^4 \mathrm{e}^{|t|(p_2(b-c)+p_1(c-a))}\big). \qquad (29)\end{aligned}$$

Case 3: *If* $\beta_2' = 0$ *and* $\alpha_2' \neq 0$, *then*

$$\begin{aligned}\Delta(\lambda) &= -\frac{2}{p_2}s^5\alpha_2'\beta_1'\delta_1\cos[p_1s(c-a)]\sin[p_2s(b-c)] + \\ &\quad + \mathrm{O}\big(|s|^4 \mathrm{e}^{|t|(p_2(b-c)+p_1(c-a))}\big), \qquad (30)\\ w(\lambda) &= -s^5\alpha_2'\beta_1'\frac{\gamma_1\delta_1}{\gamma_2 p_2}\cos[p_1s(c-a)]\sin[p_2s(b-c)] + \\ &\quad + \mathrm{O}\big(|s|^4 \mathrm{e}^{|t|(p_2(b-c)+p_1(c-a))}\big). \qquad (31)\end{aligned}$$

Case 4: *If* $\beta_2' = 0$ *and* $\alpha_2' = 0$, *then*

$$\begin{aligned}\Delta(\lambda) &= \frac{2}{p_1p_2}s^4\alpha_1'\beta_1'\delta_1\sin[p_1s(c-a)]\sin[p_2s(b-c)] + \\ &\quad + \mathrm{O}\big(|s|^3 \mathrm{e}^{|t|(p_2(b-c)+p_1(c-a))}\big), \qquad (32)\\ w(\lambda) &= s^4\alpha_1'\beta_1'\frac{\gamma_1\delta_1}{\gamma_2 p_1p_2}\sin[p_1s(c-a)]\sin[p_2s(b-c)] + \\ &\quad + \mathrm{O}\big(|s|^3 \mathrm{e}^{|t|(p_2(b-c)+p_1(c-a))}\big). \qquad (33)\end{aligned}$$

COROLLARY 10. *The eigenvalues of the problem* (1)–(5) *is bounded below.*

*Proof.* Putting $s = it$ $(t > 0)$ in the above formulas it follows that $\Delta(-t^2) \to \infty$ and $w(-t^2) \to \infty$ as $t \to \infty$. Consequently, $\Delta(\lambda) \neq 0$ and $w(\lambda) \neq 0$ for $\lambda$ negative and sufficiently large in module. □

## 4. Asymptotic Formulae for Eigenvalues

We are now ready to find the asymptotic approximation formulas for the eigenvalues of the considered problem (1)–(5).

Since the eigenvalues are coincided with the zeros of the entire functions $\Delta(\lambda)$ and $w(\lambda)$, it follows that they have no finite limits. Moreover, all eigenvalues are real and bounded below by the Corollaries 2 and 10. Therefore, we may renumber them as $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ which counted according to their multiplicity. Below we shall denote $\lambda_n = s_n^2$.

THEOREM 11. *The boundary-value-transmission problem* (1)–(5) *has an precisely denumerable many real eigenvalues, whose behaviour may be expressed by two sequence* $\{\lambda_n^{'}\}$ *and* $\{\lambda_n^{''}\}$ *with following asymptotics as* $n \to \infty$*:*

Case 1: *If* $\beta_2' \neq 0$ *and* $\alpha_2' \neq 0$*, then*

$$\begin{aligned} s_n' &= \left(\frac{\pi}{p_2(b-c)}\right)\left(n - \frac{5}{2}\right) + \mathrm{O}\left(\frac{1}{n}\right), \\ s_n'' &= \left(\frac{\pi}{p_1(c-a)}\right)\left(n + \frac{1}{2}\right) + \mathrm{O}\left(\frac{1}{n}\right). \end{aligned} \tag{34}$$

Case 2: *If* $\beta_2' \neq 0$ *and* $\alpha_2' = 0$*, then*

$$\begin{aligned} s_n' &= \left(\frac{\pi}{p_2(b-c)}\right)\left(n + \frac{1}{2}\right) + \mathrm{O}\left(\frac{1}{n}\right), \\ s_n'' &= \left(\frac{\pi}{p_1(c-a)}\right)(n-2) + \mathrm{O}\left(\frac{1}{n}\right). \end{aligned} \tag{35}$$

Case 3: *If* $\beta_2' = 0$ *and* $\alpha_2' \neq 0$*, then*

$$\begin{aligned} s_n' &= \left(\frac{\pi}{p_2(b-c)}\right)(n-2) + \mathrm{O}\left(\frac{1}{n}\right), \\ s_n'' &= \left(\frac{\pi}{p_1(c-a)}\right)\left(n + \frac{1}{2}\right) + \mathrm{O}\left(\frac{1}{n}\right). \end{aligned} \tag{36}$$

Case 4: *If* $\beta_2' = 0$ *and* $\alpha_2' = 0$*, then*

$$\begin{aligned} s_n' &= \left(\frac{\pi}{p_2(b-c)}\right)(n-2) + \mathrm{O}\left(\frac{1}{n}\right), \\ s_n'' &= \left(\frac{\pi}{p_1(c-a)}\right)n + \mathrm{O}\left(\frac{1}{n}\right). \end{aligned} \tag{37}$$

*Proof.* Let $\beta_2' \neq 0$ and $\alpha_2' \neq 0$. Denoting by $w_1(s)$ and $w_2(s)$ the first and O-term of the right of (26) respectively it is readily showed that $|w_1(s)| > |w_2(s)|$ on the contours

$$\Gamma_n' := \left\{ s' = \sigma' + it' \mid |\sigma'| \leqslant \left( \frac{\pi(n+1)}{p_2(b-c)} \right),\ |t'| \leqslant \left( \frac{\pi(n+1)}{p_2(b-c)} \right) \right\}$$

for sufficiently large $n$.

Let $\lambda_0' \leqslant \lambda_1' \leqslant \lambda_2' \leqslant \cdots$ are zeros of $w(\lambda)$, and $\lambda_n' = s_n'^2$.

Then, by applying well-known Rouche's theorem which assert that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour $\Gamma$, and $|g(s)| < |f(s)|$ on $\Gamma$, then $f(s)$ and $f(s) + g(s)$ have the same number zeros inside $\Gamma$ provided that each zeros is counted according to their multiplicity, we have

$$s_n' = \left( \frac{\pi}{p_2(b-c)} \right) \left( n - \frac{5}{2} \right) + \eta_n', \tag{38}$$

where $\eta_n' = \mathrm{O}(1)$, more precisely $|\eta_n'| \leqslant (\frac{\pi}{2p_2(b-c)})$ for sufficiently large $n$.

By putting this in (26) we get

$$\delta_1 \alpha_1' \beta_2' \cos\left[ \frac{\pi p_1(c-a)}{p_2(b-c)} \left( n - \frac{5}{2} \right) + p_1(c-a)\eta_n' \right] \times$$
$$\times \cos\left[ \pi \left( n - \frac{5}{2} \right) + p_2(b-c)\eta_n' \right] = \mathrm{O}\left( |s_n'|^{-1} \right)$$

and consequently

$$\cos\left[ \pi \left( n - \frac{5}{2} \right) + p_2(b-c)\eta_n' \right] = \mathrm{O}\left( \frac{1}{n} \right).$$

From this, since $|\eta_n'| \leqslant (\frac{\pi}{2p_2(b-c)})$, follows that $\eta_n' = \mathrm{O}(\frac{1}{n})$, which completes the proof for the first formula of the Case 1. The proofs of the other formulas are similar. □

## 5. Asymptotic Approximation Formulae for the Eigenfunctions

Let $\beta_2' \neq 0$ and $\alpha_2' \neq 0$ (Case 1).

By putting (34) in the (16) and (17) we get

$$\phi_{1\lambda_n'}(x) = \alpha_2' \left[ \left( \frac{\pi}{p_2(b-c)} \right) \left( n - \frac{5}{2} \right) \right]^2 \times$$
$$\times \cos\left[ \pi \left( n - \frac{5}{2} \right) \frac{p_1(x-a)}{p_2(b-c)} \right] + \mathrm{O}(n),$$
$$\phi_{1\lambda_n''}(x) = \alpha_2' \left[ \left( \frac{\pi}{p_1(c-a)} \right) \left( n + \frac{1}{2} \right) \right]^2 \cos\left[ \pi \left( n + \frac{1}{2} \right) \frac{(x-a)}{(c-a)} \right] + \mathrm{O}(n)$$

and

$$\begin{aligned}
\phi_{2\lambda'_n}(x) &= -\frac{1}{p_2}\alpha'_2\frac{\delta_1}{\gamma_2}\left[\left(\frac{\pi}{p_2(b-c)}\right)\left(n-\frac{5}{2}\right)\right]^3 \times \\
&\quad \times \cos\left[\pi\left(n-\frac{5}{2}\right)\frac{p_1(c-a)}{p_2(b-c)}\right]\sin\left[\pi\left(n-\frac{5}{2}\right)\frac{(x-c)}{(b-c)}\right] + \\
&\quad + \mathrm{O}(n^2), \\
\phi_{2\lambda''_n}(x) &= -\frac{1}{p_2}\alpha'_2\frac{\delta_1}{\gamma_2}\left[\left(\frac{\pi}{p_1(c-a)}\right)\left(n+\frac{1}{2}\right)\right]^3 \times \\
&\quad \times \cos\left[\pi\left(n+\frac{1}{2}\right)\right]\sin\left[\pi\left(n+\frac{1}{2}\right)\frac{p_2(x-c)}{p_1(c-a)}\right] + \mathrm{O}(n^2).
\end{aligned}$$

Hence, the eigenfunctions $\phi(x, \lambda'_n)$ and $\phi(x, \lambda''_n)$ have the following asymptotic representations for *Case 1* ($\beta'_2 \neq 0$ and $\alpha'_2 \neq 0$):

$$\phi(x,\lambda'_n) = \begin{cases}
\alpha'_2\left[\left(\dfrac{\pi}{p_2(b-c)}\right)\left(n-\dfrac{5}{2}\right)\right]^2 \cos\left[\pi\left(n-\dfrac{5}{2}\right)\dfrac{p_1(x-a)}{p_2(b-c)}\right] + \\
\quad + \mathrm{O}(n), \quad \text{for } x \in [a,c), \\
-\dfrac{1}{p_2}\alpha'_2\dfrac{\delta_1}{\gamma_2}\left[\left(\dfrac{\pi}{p_2(b-c)}\right)\left(n-\dfrac{5}{2}\right)\right]^3 \times \\
\quad \times \cos\left[\pi\left(n-\dfrac{5}{2}\right)\dfrac{p_1(c-a)}{p_2(b-c)}\right]\sin\left[\pi\left(n-\dfrac{5}{2}\right)\dfrac{(x-c)}{(b-c)}\right] + \\
\quad + \mathrm{O}(n^2), \quad \text{for } x \in (c,b],
\end{cases}$$

$$\phi(x,\lambda''_n) = \begin{cases}
\alpha'_2\left[\left(\dfrac{\pi}{p_1(c-a)}\right)\left(n+\dfrac{1}{2}\right)\right]^2 \cos\left[\pi\left(n+\dfrac{1}{2}\right)\dfrac{(x-a)}{(c-a)}\right] + \\
\quad + \mathrm{O}(n), \quad \text{for } x \in [a,c), \\
-\dfrac{1}{p_2}\alpha'_2\dfrac{\delta_1}{\gamma_2}\left[\left(\dfrac{\pi}{p_1(c-a)}\right)\left(n+\dfrac{1}{2}\right)\right]^3 \times \\
\quad \times \cos\left[\pi\left(n+\dfrac{1}{2}\right)\right]\sin\left[\pi\left(n+\dfrac{1}{2}\right)\dfrac{p_2(x-c)}{p_1(c-a)}\right] + \\
\quad + \mathrm{O}(n^2), \quad \text{for } x \in (c,b].
\end{cases}$$

Similar formulae in the other cases are as follows:

*Case 2*: If $\beta_2' \neq 0$ and $\alpha_2' = 0$, then

$$\phi(x,\lambda_n') = \begin{cases} -\dfrac{1}{p_1}\alpha_1'\left[\left(\dfrac{\pi}{p_2(b-c)}\right)\left(n+\dfrac{1}{2}\right)\right]\sin\left[\pi\left(n+\dfrac{1}{2}\right)\dfrac{p_1(x-a)}{p_2(b-c)}\right]+ \\ \quad +\mathrm{O}(1), \quad \text{for } x\in[a,c), \\ -\dfrac{1}{p_2}\alpha_1'\dfrac{\delta_1}{\gamma_2}\left[\left(\dfrac{\pi}{p_2(b-c)}\right)\left(n+\dfrac{1}{2}\right)\right]^3\times \\ \quad \times\cos\left[\pi\left(n+\dfrac{1}{2}\right)\dfrac{p_1(c-a)}{p_2(b-c)}\right]\sin\left[\pi\left(n+\dfrac{1}{2}\right)\dfrac{(x-c)}{(b-c)}\right]+ \\ \quad +\mathrm{O}(n^2), \quad \text{for } x\in(c,b], \end{cases}$$

$$\phi(x,\lambda_n'') = \begin{cases} -\dfrac{1}{p_1}\alpha_1'\left[\left(\dfrac{\pi}{p_2(b-c)}\right)\left(n+\dfrac{1}{2}\right)\right]\sin\left[\pi\left(n+\dfrac{1}{2}\right)\dfrac{p_1(x-a)}{p_2(b-c)}\right]+ \\ \quad +\mathrm{O}(1), \quad \text{for } x\in[a,c), \\ -\dfrac{1}{p_2}\alpha_1'\dfrac{\delta_1}{\gamma_2}\left[\left(\dfrac{\pi}{p_2(b-c)}\right)\left(n+\dfrac{1}{2}\right)\right]^3\times \\ \quad \times\cos\left[\pi\left(n+\dfrac{1}{2}\right)\dfrac{p_1(c-a)}{p_2(b-c)}\right]\sin\left[\pi\left(n+\dfrac{1}{2}\right)\dfrac{(x-c)}{(b-c)}\right]+ \\ \quad +\mathrm{O}(n^2), \quad \text{for } x\in(c,b]. \end{cases}$$

*Case 3*: If $\beta_2' = 0$ and $\alpha_2' \neq 0$, then

$$\phi(x,\lambda_n') = \begin{cases} \alpha_2'\left[\left(\dfrac{\pi}{p_2(b-c)}\right)(n-2)\right]^2\cos\left[\pi(n-2)\dfrac{p_1(x-a)}{p_2(b-c)}\right]+ \\ \quad +\mathrm{O}(n), \quad \text{for } x\in[a,c), \\ -\dfrac{1}{p_2}\alpha_2'\dfrac{\delta_1}{\gamma_2}\left[\left(\dfrac{\pi}{p_2(b-c)}\right)(n-2)\right]^3\times \\ \quad \times\cos\left[\pi(n-2)\dfrac{p_1(c-a)}{p_2(b-c)}\right]\sin\left[\pi(n-2)\dfrac{(x-c)}{(b-c)}\right]+ \\ \quad +\mathrm{O}(n^2), \quad \text{for } x\in(c,b], \end{cases}$$

$$\phi(x,\lambda_n'') = \begin{cases} \alpha_2'\left[\left(\dfrac{\pi}{p_1(c-a)}\right)\left(n+\dfrac{1}{2}\right)\right]^2\cos\left[\pi\left(n+\dfrac{1}{2}\right)\dfrac{(x-a)}{(c-a)}\right]+ \\ \quad +\mathrm{O}(n), \quad \text{for } x\in[a,c), \\ \mathrm{O}(n^2), \quad \text{for } x\in(c,b]. \end{cases}$$

*Case 4*: If $\beta_2' = 0$ and $\alpha_2' = 0$, then

$$\phi(x,\lambda_n') = \begin{cases} \alpha_1' \cos\left[\pi(n-2)\dfrac{p_1(x-a)}{p_2(b-c)}\right] + \mathrm{O}(n), \quad \text{for } x \in [a,c), \\ -\dfrac{1}{p_2}\alpha_1'\dfrac{\delta_1}{\gamma_2}\left[\left(\dfrac{\pi}{p_2(b-c)}\right)(n-2)\right]^3 \times \\ \quad \times \cos\left[\pi(n-2)\dfrac{p_1(c-a)}{p_2(b-c)}\right]\sin\left[\pi(n-2)\dfrac{(x-c)}{(b-c)}\right] + \\ \quad + \mathrm{O}(n^2), \quad \text{for } x \in (c,b], \end{cases}$$

$$\phi(x,\lambda_n'') = \begin{cases} \alpha_1'\left[\dfrac{\pi n}{p_1(c-a)}\right]^2 \cos\left[\pi n\dfrac{(x-a)}{(c-a)}\right] + \mathrm{O}(n), \quad \text{for } x \in [a,c), \\ (-1)^{n+1}\dfrac{1}{p_2}\alpha_1'\dfrac{\delta_1}{\gamma_2}\left[\dfrac{\pi n}{p_1(c-a)}\right]^3 \sin\left[\pi n\dfrac{p_2(x-c)}{p_1(c-a)}\right] + \\ \quad + \mathrm{O}(n^2), \quad \text{for } x \in (c,b]. \end{cases}$$

All this asymptotic approximations are hold uniformly for $x$.

*Remark.* We can show that the solutions $\phi(x,\lambda_0)$ and $\chi(x,\lambda_0)$ are linearly dependent in the case when $\lambda_0$ is an eigenvalue. Therefore, the asymptotic approximations of $\chi(x,\lambda_n')$ and $\chi(x,\lambda_n'')$ do not require individual consideration.

## References


1. Binding, P. A., Browne, P. J. and Watson, B. A.: Strum–Liouville problems with boundary conditions rationally dependent on the eigenparameter II, *J. Comput. Appl. Math.* **148** (2002), 147–169.
2. Fulton, C. T.: Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh A* **77** (1977), 293–308.
3. Kerimov, N. B. and Memedov, Kh. K.: On a boundary value problem with a spectral parameter in the boundary conditions, *Sibirsk. Math. Zh.* **40**(2) (1999), 325–335. English translation: *Siberian Math. J.* **40**(2) (1999), 281–290.
4. Likov, A. V. and Mikhailov, Yu. A.: *The Theory of Heat and Mass Transfer*, Gosenergoizdat, 1963 (Russian).
5. Mukhtarov, O. Sh.: Discontinuous boundary-value problem with spectral parameter in boundary conditions, *Turkish J. Math.* **18** (1994), 183–192.
6. Mukhtarov, O. Sh. and Demir, H.: Coerciveness of the discontinuous initial-boundary value problem for parabolic equations, *Israel J. Math.* **114** (1999), 239–252.
7. Mukhtarov, O. Sh., Kandemir, M. and Kuruoğlu, N.: Distribution of eigenvalues for the discontinunous boundary value problem with functional manypoint conditions, *Israel J. Math.* **129** (2002), 143–156.
8. Mukhtarov, O. Sh. and Yakubov, S.: Problems for ordinary differential equations with transmission conditions, *Appl. Anal.* **81** (2002), 1033–1064.
9. Rasulov, M. L.: *Metods of Contour Integratian*, North-Holland, Amsterdam, 1967.
10. Rasulov, M. L.: *Application of the Contour Integral Method*, Nauka, Moscow, 1975 (Russian).

11. Shkalikov, A. A.: Boundary value problems for ordinary differential equations with a parameter in boundary conditions, *Trudy Sem. Petrovsk.* **9** (1983), 190–229 (Russian).
12. Tikhonov, A. N. and Samarskii, A. A.: *Equations of Mathematical Physics*, Pergamon, Oxford, 1963.
13. Titchmarsh, E. C.: *Eigenfunction Expensions Associated with Second Order Differential Equations I*, 2nd edn, Oxford Univ. Press, London, 1962.
14. Titeux, I. and Yakubov, Ya.: Completeness of root functions for thermal conduction in a strip with piecewise continuous coefficients, *Math. Models Methods Appl. Sci.* **7**(7) (1997), 1035–1050.
15. Voitovich, N. N., Katsenelbaum, B. Z. and Sivov, A. N.: *Generalized Method of Eigen-Vibration in the Theory of Diffraction*, Nauka, Moscow, 1997 (Russian).
16. Yakubov, S.: *Completeness of Root Functions of Regular Differential Operators*, Logman, Scientific and Technical, New York, 1994.
17. Yakubov, S. and Yakubov, Y.: Abel basis of root functions of regular boundary value problems, *Math. Nachr.* **197** (1999), 157–187.
18. Yakubov, S. and Yakubov, Y.: *Differential-Operator Equations. Ordinary and Partial Differential Equations*, Chapman and Hall/CRC, Boca Raton, 2000.