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Oscillation of second-order noncanonical neutral differential equations with distributed deviating arguments

Ercan Tunç[1] · Kaşif Baş[2] · Orhan Özdemir[1] · Ethiraju Thandapani[3]

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Abstract
This study is devoted to the oscillatory behaviour of all solutions to a class of second-order noncanonical differential equations of neutral type with distributed deviating arguments. Some new sufficient conditions for all solutions of the equation considered here to be oscillatory are presented. Examples showing the applicability of the main results are established.

Keywords Noncanonical equations · Differential equations · Second-order · Neutral · Oscillation

Mathematics Subject Classification 34K40 · 34K11 · 34C10

1 Introduction

In this paper, we study the oscillatory behaviour of solutions of the second-order noncanonical neutral differential equation with distributed deviating arguments

$$\left[a(t)\left[u(t)+p(t)u(\vartheta(t))\right]'\right]'+\int_{\alpha}^{\beta}q(t,\mu)u^{\delta}(\phi(t,\mu))d\mu=0, \tag{EQ}$$

where $t\geq t_0>0$, δ is the ratio of positive odd integers with $0<\delta\leq 1$ and $0<\alpha<\beta<\infty$. We suppose throughout this study that the following assumptions hold:

✉ Orhan Özdemir
orhanozdemir37@yahoo.com

Ercan Tunç
ercantunc72@yahoo.com

Kaşif Baş
kasifbas@gmail.com

Ethiraju Thandapani
ethandapani@yahoo.co.in

[1] Department of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpasa University, 60240 Tokat, Turkey

[2] Institute of Graduate Studies, Tokat Gaziosmanpasa University, 60240 Tokat, Turkey

[3] Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600005, India

(i) $a \in C([t_0, \infty), (0, \infty))$, $p \in C([t_0, \infty), \mathbb{R})$ with $p(t) \geq \ell > 1$;
(ii) $q \in C([t_0, \infty) \times [\alpha, \beta], [0, \infty))$ and $q(t, \mu) \not\equiv 0$ on any interval of the form $[t_*, \infty) \times [\alpha, \beta]$, $t_* \geq t_0$;
(iii) $\vartheta \in C([t_0, \infty), \mathbb{R})$ is strictly increasing, $\vartheta(t) \leq t$, and $\lim_{t\to\infty} \vartheta(t) = \infty$;
(iv) $\phi \in C([t_0, \infty) \times [\alpha, \beta], \mathbb{R})$ is nonincreasing in its second variable and nondecreasing in its first variable, and $\lim_{t\to\infty} \phi(t, \mu) = \infty$, $\mu \in [\alpha, \beta]$.

We let

$$I(t) = \int_t^\infty \frac{1}{a(s)} ds, \quad t \geq t_0,$$

and assume that Eq. (EQ) is in noncanonical form, i.e.,

$$I(t_0) < \infty, \tag{1.1}$$

and we wish to note that Eq. (EQ) is said to be in canonical form if

$$I(t_0) = \infty. \tag{1.2}$$

By a *solution* of Eq. (EQ), we understand a continuous function $u : [t_u, \infty) \to \mathbb{R}$ for some $t_u \geq t_0$ such that $u(t) + p(t)u(\vartheta(t)) \in C^1([t_u, \infty), \mathbb{R})$, $a(t)[u(t) + p(t)u(\vartheta(t))]' \in C^1([t_u, \infty), \mathbb{R})$, and u satisfies (EQ) on $[t_u, \infty)$.

Since we are only interested in the oscillatory behaviour of solutions, every solution $u(t)$ of (EQ) considered here is assumed to be continuable and nontrivial, i.e., $u(t)$ is defined on $[t_u, \infty)$ for some $t_u \geq t_0$ and $\sup\{|u(t)| : t_1 \leq t < \infty\} > 0$ for any $t_1 \geq t_u$. We make a standing hypothesis that (EQ) does possess such solutions. Such a solution is said to be oscillatory if its set of zeros is unbounded above and is said to be nonoscillatory otherwise. Equation (EQ) is said to be oscillatory if all its nontrivial continuable solutions are oscillatory.

Functional differential equations of neutral type are equations in which the highest-order derivative of the unknown function appears in the equation both without and with deviating arguments. Equations of this type continue to attract the attention of researchers since they have applications in technology and natural sciences, as well as attracting the attention of researchers from a theoretical point of view. Readers can refer to the monographs (Györi and Ladas 1991; Hale 1977) and the paper (Brayton and Willoughby 1967) for some of the most popular applications of neutral differential equations.

The problem of determining oscillation criteria for various classes of second order differential equations of neutral type without distributed deviating arguments has received a great deal of attention by researchers over the years; see, e.g., Agarwal et al. (2014, 2016), Bohner et al. (2017, 2019, 2020), Dong (2010), Grace et al. (2018, 2021, 2023), Grace and Graef (2018), Jadlovská (2021), Li et al. (2013a, b), Li and Rogovchenko (2015, 2017), Prabaharan et al. (2019), Saker et al. (2007), Sethi and Tripathy (2021), Sui and Han (2018), Tamilvanan et al. (2017), Tripathy and Santra (2021), Xu and Meng (2006) and the references cited therein for some interesting results and more details. However, from the review of literature, it is clear that far less is known about the oscillatory behaviour of second order differential equations of neutral type with distributed deviating arguments; we refer the reader to Candan (2011, 2015), Grace et al. (2014, 2017), Graef et al. (2021), Gui and Xu (2007), Li and Thandapani (2011), Qi and Yu (2015), Thandapani and Piramanantham (2010), Tunç and Graef (2014), Wang (2004), Xu and Weng (2007), Yang et al. (2016a, b), Zhao and Meng (2008) and the references contained therein. We believe that it would be useful to present a summary of the equations of this type discussed in the literature so far in order to present the distinctive features of the presented paper more clearly and to create a literature that can

be easily accessed by the reader interested in the oscillation properties of the equations of neutral type.

Wang (2004) investigated the oscillatory behaviour of solutions of the equation

$$\left[a(t)\left(u(t)+p(t)u(t-\vartheta)\right)'\right]'+\int_{\alpha}^{\beta}q(t,\mu)u(\phi(t,\mu))d\sigma(\mu)=0, \tag{1.3}$$

under condition (1.2), where $\vartheta>0$ is a constant and $0\le p(t)<1$.

Thandapani and Piramanantham (2010) studied the oscillation of all solutions of the equation

$$\left[a(t)\left(u(t)+p(t)u(t-\vartheta)\right)'\right]'+\int_{\alpha}^{\beta}q(t,\mu)f\left(u(\phi(t,\mu))\right)d\mu=0, \tag{1.4}$$

under conditions (1.1) and (1.2), where $\vartheta>0$ is a constant, $0\le p(t)<1$, and $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $zf(z)$ is positive for $z\neq 0$, and $f(-z)=-f(z)$.

Gui and Xu (2007), Xu and Weng (2007) and Zhao and Meng (2008) discussed the oscillation of the equation

$$\left[a(t)\Phi(u(t))\left(u(t)+p(t)u(t-\vartheta)\right)'\right]'+\int_{\alpha}^{\beta}q(t,\mu)f\left(u(\phi(t,\mu))\right)d\sigma(\mu)=0, \tag{1.5}$$

under condition (1.2), where $\vartheta>0$ is a constant, $0\le p(t)<1$, $\Phi\in C^1(\mathbb{R},\mathbb{R})$ with $\Phi(u)$ is positive for $u\neq 0$, and $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $zf(z)$ is positive for $z\neq 0$, and f satisfies $f(u)/u\ge L$ with $L>0$ or $f'(u)\ge M$ with $M>0$.

Li and Thandapani (2011) examined the oscillatory behaviour of all solutions of the equation

$$\left[a(t)\left(\left(u(t)+p(t)u(\vartheta(t))\right)'\right)^{\delta}\right]'+\int_{\alpha}^{\beta}q(t,\mu)u^{\delta}(\phi(t,\mu))d\mu=0, \tag{1.6}$$

under condition

$$\int_{t_0}^{\infty}\frac{1}{a^{1/\delta}(s)}ds=\infty, \tag{1.7}$$

where $\delta\ge 1$ is a quotient of positive odd integers, $0\le p(t)<a_0$ with a_0 is a constant, and $\vartheta\in C^1([t_0,\infty),\mathbb{R})$, $\vartheta'(t)\ge\vartheta_0>0$, $\vartheta([t_0,\infty))=[\vartheta(t_0),\infty)$, and $\vartheta[\phi(t,\mu)]=\phi[\vartheta(t),\mu]$.

Candan (2011) and Yang et al. (2016a) studied the oscillation of the equation

$$\left[a(t)\left(\left(u(t)+p(t)u(\vartheta(t))\right)'\right)^{\delta}\right]'+\int_{\alpha}^{\beta}f\left(t,u(\phi(t,\mu))\right)d\mu=0, \tag{1.8}$$

under condition (1.7), where δ is the quotient of two odd positive integers, $0\le p(t)<1$, $f:[t_0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $zf(t,z)$ is positive for $z\neq 0$; and in Candan (2011) it is assumed that there exists a positive valued function $q\in C([t_0,\infty),\mathbb{R})$ such that $|f(t,z)|\ge q(t)|z^{\theta}|$ with θ is the ratio of odd natural numbers, and in Yang et al. (2016a) it is assumed that there exists a function $q(t,\mu)\in([t_0,\infty)\times[\alpha,\beta],\mathbb{R}^+)$ such that $|f(t,z)|\ge q(t,\mu)|z^{\delta}|$.

Yang et al. (2016b) investigated

$$\left[a(t)|\psi'(t)|^{\delta-1}\psi'(t)\right]'+\int_{\alpha}^{\beta}q(t,\mu)|u(\phi(t,\mu))|^{\theta-1}u(\phi(t,\mu))d\mu=0, \tag{1.9}$$

under condition (1.7), where $\psi(t) = u(t) + p(t)u(\vartheta(t))$, δ and θ are positive constants, and $0 \leq p(t) < 1$.

Tunç and Graef (2014) studied the oscillatory and asymptotic behaviour of the equation

$$\left[a(t)|\psi'(t)|^{\delta-1}\psi'(t)\right]' + \int_{\alpha}^{\beta} f\left(t, u(\phi(t,\mu))\right) d\mu = 0, \tag{1.10}$$

under the conditions (1.7) and

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\delta}(s)} ds < \infty,$$

where $\psi(t) = u(t) + p(t)u(\vartheta(t))$, $\delta \geq 1$ is a constant, $0 \leq p(t) < 1$, $f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $zf(t,z)$ is positive for $z \neq 0$, and there exists a function $q \in C([t_0, \infty), \mathbb{R}^+)$ such that $|f(t,z)| \geq q(t)|z^{\theta}|$ with $\theta \in (0,1)$ is the quotient of odd positive integers.

In the light of the above observations, the studies mentioned above (except Graef et al. 2021, where an equation with a damping term in canonical form was considered) were considered under assumption of the neutral coefficient function $p(t)$ is bounded. Therefore, these studies cannot be applied to the case where $p(t) \to \infty$ as $t \to \infty$. Based on this reality, the main objective of this study is to build several novel oscillation theorems that can be applied not only to the case when $p(t) \to \infty$ as $t \to \infty$ but also to the case when $p(t)$ is constant or bounded. We also ask to mention that the results presented here are novel even for $\delta = 1$ and discrete delay such as $\vartheta(t) = t - \vartheta$ with ϑ is a positive constant. However, our results here cannot be applied to the case where $0 \leq p(t) < 1$ and this remains the weak side of our results for now. We also wish to express that the results presented here are mainly motivated by the results of Grace et al. (2022) and Koplatadze et al. (1999). As can be seen from the equation types mentioned above, it would be possible to extend this paper's new criteria to the more general equations; for instance, the equations of type studied in Candan (2011), Gui and Xu (2007), Li and Thandapani (2011), Tunç and Graef (2014), Xu and Weng (2007), Yang et al. (2016a, b), Zhao and Meng (2008) and to the different types that involve Eq. (EQ) as a specific case. On account of this, it is hoped that this current study will stimulate additional interest in research on functional differential equations of neutral type in general, and those with distributed deviating arguments and unbounded neutral coefficients especially.

Throughout this study, we suppose that all functional inequalities hold for all sufficiently large t. Due to if $u(t)$ is a solution of (EQ), then $-u(t)$ is also a solution, we will deal only with the eventually positive solutions of (EQ).

2 Main results

For convenience of the reader, we list here some notations as below:

$$\psi(t) := u(t) + p(t)u(\vartheta(t)),\ A(t) := \int_{t_*}^{t} \frac{1}{a(s)} ds \text{ for } t \geq t_* \geq t_0,$$

$$\phi_1(t) := \phi(t,\beta),\ \phi_2(t) := \phi(t,\alpha),\ g(t) := \vartheta^{-1}(\phi_1(t)),\ h(t) := \vartheta^{-1}(\phi_2(t)),$$

$$\pi(t) := \frac{1}{p(\vartheta^{-1}(t))}\left[1 - \frac{1}{p(\vartheta^{-1}(\vartheta^{-1}(t)))}\right],$$

$$q_1(t) := \int_\alpha^\beta q(t,\mu)p^{-\delta}(\vartheta^{-1}(\phi(t,\mu)))d\mu, \quad \text{and} \quad q_2(t) := \int_\alpha^\beta q(t,\mu)\pi^\delta(\phi(t,\mu))d\mu,$$

where ϑ^{-1} is the inverse function of ϑ.

We first present the following three oscillation criteria for Eq. (EQ) in the case where $\phi_2(t) \le \vartheta(t)$, i.e., h is a delay argument.

Theorem 2.1 *Let* (1.1) *hold and let* $\phi_2(t) \le \vartheta(t)$. *If*

$$\int_{t_0}^\infty I(s)q_1(s)ds = \infty \tag{2.1}$$

and

$$\limsup_{t\to\infty}\left(I(t)\int_{t_0}^t q_2(s)ds + I^{-\delta}(h(t))\int_t^\infty I(s)q_2(s)I^\delta(h(s))ds\right) > \begin{cases}1 & \text{if } \delta = 1,\\ 0 & \text{if } \delta \in (0,1),\end{cases} \tag{2.2}$$

then Eq. (EQ) is oscillatory.

Proof Suppose to the contrary that the function $u(t)$ be a nonoscillatory (say eventually positive) solution of Eq. (EQ). Then $u(t) > 0$ and $u(\vartheta(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$ and $u(\phi(t,\mu)) > 0$ for $(t,\mu) \in [t_1,\infty)\times[\alpha,\beta]$. From the definition of ψ, we see that $0 < u(t) \le \psi(t)$ and

$$\begin{aligned} u(t) &= \frac{1}{p(\vartheta^{-1}(t))}\left[\psi(\vartheta^{-1}(t)) - u(\vartheta^{-1}(t))\right]\\ &\ge \frac{\psi(\vartheta^{-1}(t))}{p(\vartheta^{-1}(t))} - \frac{\psi(\vartheta^{-1}(\vartheta^{-1}(t)))}{p(\vartheta^{-1}(t))p(\vartheta^{-1}(\vartheta^{-1}(t)))}. \end{aligned} \tag{2.3}$$

By (iii), we see that ϑ^{-1} is increasing and furthermore $t \le \vartheta^{-1}(t)$. Hence,

$$\vartheta^{-1}(t) \le \vartheta^{-1}(\vartheta^{-1}(t)). \tag{2.4}$$

It follows from (EQ) that

$$\left(a(t)\psi'(t)\right)' = -\int_\alpha^\beta q(t,\mu)u^\delta(\phi(t,\mu))d\mu \le 0 \tag{2.5}$$

for $t \ge t_1$, which indicates that $a(t)\psi'(t)$ is nonincreasing and eventually of one sign. That is, a $t_2 \ge t_1$ can be found such that, either

$(\mathcal{N}_1)$ $\psi'(t) > 0$, $\left(a(t)\psi'(t)\right)' \le 0$, or
$(\mathcal{N}_0)$ $\psi'(t) < 0$, $\left(a(t)\psi'(t)\right)' \le 0$

for $t \ge t_2$. We now consider case $(\mathcal{N}_1)$. Then

$$\psi(t) \ge \int_{t_2}^t \frac{a(s)\psi'(s)}{a(s)}ds \ge \left(\int_{t_2}^t \frac{1}{a(s)}ds\right)a(t)\psi'(t) = A(t)\psi'(t)a(t),$$

which yields, for $t \geq t_3$ for some $t_3 > t_2$,

$$\left(\frac{\psi(t)}{A(t)}\right)' = \frac{\psi'(t)a(t)A(t) - \psi(t)}{a(t)A^2(t)} \leq 0,$$

i.e., $\psi(t)/A(t)$ is nonincreasing. From this and (2.4), we observe that

$$\psi\left(\vartheta^{-1}(\vartheta^{-1}(t))\right) \leq \frac{A\left(\vartheta^{-1}(\vartheta^{-1}(t))\right)\psi(\vartheta^{-1}(t))}{A\left(\vartheta^{-1}(t)\right)}. \tag{2.6}$$

Using (2.6) in (2.3) yields

$$u(t) \geq \frac{\psi(\vartheta^{-1}(t))}{p(\vartheta^{-1}(t))}\left[1 - \frac{1}{p(\vartheta^{-1}(\vartheta^{-1}(t)))}\frac{A(\vartheta^{-1}(\vartheta^{-1}(t)))}{A(\vartheta^{-1}(t))}\right] \text{ for } t \geq t_3. \tag{2.7}$$

Using the fact that

$$\lim_{t\to\infty}\frac{A(\vartheta^{-1}(t))}{A(t)} = 1,$$

and taking $p(t) \geq \ell > 1$ into account, we see that there exists $t_3 \geq t_2$ such that for any $\epsilon \in (0, \ell - 1)$,

$$\frac{A(\vartheta^{-1}(\vartheta^{-1}(t)))}{A(\vartheta^{-1}(t))}\frac{1}{p(\vartheta^{-1}(\vartheta^{-1}(t)))} \leq \frac{1+\epsilon}{\ell}$$

holds for $t \geq t_3$. Using this in (2.7) gives

$$u(t) \geq d\frac{\psi(\vartheta^{-1}(t))}{p(\vartheta^{-1}(t))} \text{ for } t \geq t_3, \tag{2.8}$$

where $d = 1 - (1+\epsilon)/\ell > 0$. Due to $\lim_{t\to\infty}\phi(t,\mu) = \infty$, a $t_4 \geq t_3$ can be chosen such that $\phi(t,\mu) \geq t_3$ for all $t \geq t_4$. Thence, it leads from (2.8) that

$$u(\phi(t,\mu)) \geq d\frac{\psi(\vartheta^{-1}(\phi(t,\mu)))}{p(\vartheta^{-1}(\phi(t,\mu)))} \text{ for } t \geq t_4. \tag{2.9}$$

Substituting (2.9) into (EQ) gives

$$\left(a(t)\psi'(t)\right)' + d^\delta\int_\alpha^\beta q(t,\mu)p^{-\delta}(\vartheta^{-1}(\phi(t,\mu)))\psi^\delta\left(\vartheta^{-1}(\phi(t,\mu))\right)d\mu \leq 0. \tag{2.10}$$

Since ϕ is nonincreasing in μ, and ϑ and ψ are increasing, we deduce from (2.10) that

$$\left(a(t)\psi'(t)\right)' + d^\delta\left(\int_\alpha^\beta q(t,\mu)p^{-\delta}(\vartheta^{-1}(\phi(t,\mu)))d\mu\right)\psi^\delta\left(\vartheta^{-1}(\phi_1(t))\right) \leq 0,$$

or

$$\left(a(t)\psi'(t)\right)' + d^\delta q_1(t)\psi^\delta(g(t)) \leq 0 \text{ for } t \geq t_4. \tag{2.11}$$

Since $\psi(t) > 0$ and $\psi'(t) > 0$, for some $c > 0$, $\psi(t) \geq c > 0$ for $t \geq t_4$ and so inequality (2.11) takes the form

$$\left(a(t)\psi'(t)\right)' + c^\delta d^\delta q_1(t) \leq 0 \tag{2.12}$$

for $t \geq t_5$ for some $t_5 \geq t_4$. Integrating inequality (2.12) twice yields

$$\psi(t) \leq \psi(t_5) + \int_{t_5}^t \frac{a(t_5)\psi'(t_5)}{a(s)}ds - c^\delta d^\delta\int_{t_5}^t\frac{1}{a(z)}\int_{t_5}^z q_1(s)dsdz \to -\infty$$

as $t \to \infty$ due to (1.1) and (2.1). This is a contradiction with the positivity of $\psi(t)$.

Now, let us consider case $(\mathcal{N}_0)$. Using the fact that $\psi'(t) < 0$, it follows from (2.4) that

$$\psi(\vartheta^{-1}(t)) \geq \psi(\vartheta^{-1}(\vartheta^{-1}(t))). \tag{2.13}$$

Using (2.13) in (2.3) leads to

$$u(t) \geq \frac{1}{p(\vartheta^{-1}(t))}\left[1 - \frac{1}{p(\vartheta^{-1}(\vartheta^{-1}(t)))}\right]\psi(\vartheta^{-1}(t)) \text{ for } t \geq t_2. \tag{2.14}$$

By (i), we see that

$$u(t) \geq b\frac{\psi(\vartheta^{-1}(t))}{p(\vartheta^{-1}(t))} \text{ for } t \geq t_2, \tag{2.15}$$

where $b = 1 - 1/\ell > 0$, and so

$$u(\phi(t,\mu)) \geq b\frac{\psi(\vartheta^{-1}(\phi(t,\mu)))}{p(\vartheta^{-1}(\phi(t,\mu)))}, \tag{2.16}$$

for $t \geq t_3$ for some $t_3 \geq t_2$. Substituting (2.16) into Eq. (EQ) gives

$$\left(a(t)\psi'(t)\right)' + b^\delta \int_\alpha^\beta q(t,\mu)p^{-\delta}(\vartheta^{-1}(\phi(t,\mu)))\psi^\delta\left(\vartheta^{-1}(\phi(t,\mu))\right)d\mu \leq 0. \tag{2.17}$$

Since ϑ is increasing, ψ is decreasing, and ϕ is nonincreasing in μ, we deduce from (2.17) that

$$\left(a(t)\psi'(t)\right)' + b^\delta\left(\int_\alpha^\beta q(t,\mu)p^{-\delta}(\vartheta^{-1}(\phi(t,\mu)))d\mu\right)\psi^\delta\left(\vartheta^{-1}(\phi_2(t))\right) \leq 0,$$

or

$$\left(a(t)\psi'(t)\right)' + b^\delta q_1(t)\psi^\delta(h(t)) \leq 0 \text{ for } t \geq t_3. \tag{2.18}$$

Since ψ is positive and decreasing, there exists a fixed $0 \leq \kappa < \infty$ such that

$$\lim_{t\to\infty}\psi(t) = \kappa.$$

We claim that $\kappa = 0$. If not, then there exist $t_4 \geq t_3$ such that

$$\psi(t) > \kappa. \tag{2.19}$$

Using (2.19) in (2.18) yields

$$\left(a(t)\psi'(t)\right)' + b^\delta\kappa^\delta q_1(t) \leq 0 \tag{2.20}$$

for $t \geq t_5$ for some $t_5 \geq t_4$. Integrating (2.20) from t_5 to t gives

$$-a(t)\psi'(t) \geq b^\delta\kappa^\delta\int_{t_5}^t q_1(s)ds. \tag{2.21}$$

Integrating (2.21) from t_5 to t and taking (2.1) into account, one can observe that

$$\psi(t_5) \geq b^\delta\kappa^\delta\int_{t_5}^t\frac{1}{a(z)}\int_{t_5}^z q_1(s)dsdz \geq b^\delta\kappa^\delta\int_{t_5}^t\int_s^t\frac{1}{a(z)}q_1(s)dzds.$$

Letting $t \to \infty$ in the above inequality, we obtain a contradiction to (2.1), and so

$$\lim_{t\to\infty}\psi(t) = 0.$$

By case $(\mathcal{N}_0)$, we see that

$$\psi(t) \geq -\int_t^\infty \frac{a(s)\psi'(s)}{a(s)}ds \geq -\left(\int_t^\infty \frac{1}{a(s)}ds\right) a(t)\psi'(t) = -I(t)a(t)\psi'(t),$$

for $t \geq t_2$, and hence

$$\psi(t) + a(t)I(t)\psi'(t) \geq 0. \tag{2.22}$$

From (2.22), we observe that

$$\left(\frac{\psi(t)}{I(t)}\right)' = \frac{a(t)I(t)\psi'(t) + \psi(t)}{a(t)I^2(t)} \geq 0,$$

that is, $\psi(t)/I(t)$ is eventually nondecreasing, say for $t \geq t_3$ for some $t_3 \geq t_2$. From (2.14) and (EQ), we observe that

$$\big(a(t)\psi'(t)\big)' + q_2(t)\psi^\delta(h(t)) \leq 0 \text{ for } t \geq t_3, \tag{2.23}$$

which can be expressed in an equivalent form as below

$$(a(t)\psi'(t)I(t) + \psi(t))' + I(t)q_2(t)\psi^\delta(h(t)) \leq 0. \tag{2.24}$$

Integrating (2.24) from $t \geq t_3$ to z, letting $z \to \infty$, and using (2.22) gives

$$a(t)I(t)\psi'(t) + \psi(t) \geq \int_t^\infty I(s)q_2(s)\psi^\delta(h(s))ds. \tag{2.25}$$

Integrating (2.23) from t_3 to t and then multiplying by $I(t)$ leads to

$$-a(t)\psi'(t)I(t) \geq I(t)\int_{t_3}^t q_2(s)\psi^\delta(h(s))ds. \tag{2.26}$$

From (2.25) and (2.26), we deduce that

$$\psi(t) \geq I(t)\int_{t_3}^t q_2(s)\psi^\delta(h(s))ds + \int_t^\infty I(s)q_2(s)\psi^\delta(h(s))ds. \tag{2.27}$$

From the fact that h is increasing and ψ is decreasing, we get $h(s) \leq h(t) \leq t$ for $s \leq t$, and hence $\psi(t) \leq \psi(h(t)) \leq \psi(h(s))$ for $t \geq t_3$. From these, we obtain

$$\int_{t_3}^t q_2(s)\psi^\delta(h(s))ds \geq \left(\int_{t_3}^t q_2(s)ds\right)\psi^\delta(t). \tag{2.28}$$

On the other hand, for $s > t$, we get $h(s) > h(t)$ and $h(t) \leq t$. In view of the fact that $\psi(t)/I(t)$ is nondecreasing, we see that

$$\frac{\psi^\delta(h(s))}{I^\delta(h(s))} \geq \frac{\psi^\delta(h(t))}{I^\delta(h(t))} \geq \frac{\psi^\delta(t)}{I^\delta(h(t))}.$$

Thus,

$$\begin{aligned}\int_t^\infty I(s)q_2(s)\psi^\delta(h(s))ds &\geq \int_t^\infty I(s)q_2(s)I^\delta(h(s))\frac{\psi^\delta(h(s))}{I^\delta(h(s))}ds \\ &\geq \left(\frac{1}{I^\delta(h(t))}\int_t^\infty I(s)q_2(s)I^\delta(h(s))ds\right)\psi^\delta(t).\end{aligned} \tag{2.29}$$

Using (2.28) and (2.29) in (2.27), we arrive at

$$\psi(t) \geq \left(I(t) \int_{t_3}^{t} q_2(s)ds \right) \psi^{\delta}(t) + \left(\frac{1}{I^{\delta}(h(t))} \int_{t}^{\infty} I(s)q_2(s)I^{\delta}(h(s))ds \right) \psi^{\delta}(t).$$

Letting lim sup as $t \to \infty$ of the resulting inequality, we get a contradiction to (2.2), which completes the theorem's proof. □

Theorem 2.2 *Let* (1.1) *and* (2.1) *be satisfied and let* $\phi_2(t) \leq \vartheta(t)$. *If*

$$\limsup_{t\to\infty} \left(I(h(t)) \int_{t_0}^{h(t)} q_2(s)ds + \int_{h(t)}^{t} I(s)q_2(s)ds \right.$$
$$\left. + I^{-\delta}(h(t)) \int_{t}^{\infty} I(s)q_2(s)I^{\delta}(h(s))ds \right) > \begin{cases} 1 & \text{if } \delta = 1, \\ 0 & \text{if } \delta \in (0,1), \end{cases} \tag{2.30}$$

then Eq. (EQ) oscillates.

Proof Proceeding as in the proof of Theorem 2.1, we again arrive at (2.27), which takes the form

$$\psi(h(t)) \geq I(h(t)) \int_{t_3}^{h(t)} q_2(s)\psi^{\delta}(h(s))ds + \int_{h(t)}^{t} I(s)q_2(s)\psi^{\delta}(h(s))ds$$
$$+ \int_{t}^{\infty} I(s)q_2(s)\psi^{\delta}(h(s))ds.$$

The rest of the proof is similar to that of Theorem 2.1 and so details are skipped. The proof is complete. □

Theorem 2.3 *Let* (1.1) *and* (2.1) *be satisfied and let* $\phi_2(t) \leq \vartheta(t)$. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} I(s)q_2(s)ds > \begin{cases} 1 & \text{if } \delta = 1, \\ 0 & \text{if } \delta \in (0,1), \end{cases} \tag{2.31}$$

then Eq. (EQ) oscillates.

Proof Proceeding as in the proof of Theorem 2.1, we again arrive at (2.22) and (2.24). Setting

$$\omega(t) := \psi(t) + I(t)a(t)\psi'(t),$$

we have $0 \leq \omega(t) \leq \psi(t)$, and so it follows from (2.24) that

$$\omega'(t) + I(t)q_2(t)\omega^{\delta}(h(t) \leq 0.$$

If this last inequality is integrated from $h(t)$ to t, it can be seen that

$$\omega(h(t)) \geq \int_{h(t)}^{t} I(s)q_2(s)\omega^{\delta}(h(s))ds \geq \omega^{\delta}(h(t)) \int_{h(t)}^{t} I(s)q_2(s)ds,$$

which yields

$$\omega^{1-\delta}(h(t)) \geq \int_{h(t)}^{t} I(s)q_2(s)ds.$$

Taking the lim sup as $t \to \infty$ on both sides of the above inequality, we get a conclusion that contradicts with condition (2.31), and we complete the proof. □

Next, we present the following two oscillation criteria for Eq. (EQ) in the case where $\phi_1(t) \geq \vartheta(t)$, i.e., h is an advanced argument.

Theorem 2.4 *Let (1.1) and (2.1) are satisfied and let $\phi_1(t) \geq \vartheta(t)$. If*

$$\limsup_{t\to\infty}\left(\frac{I^{\delta}(h(t))}{I^{\delta-1}(t)}\int_{t_0}^{t} q_2(s)ds + \frac{1}{I^{\delta}(t)}\int_{t}^{\infty} I(s)q_2(s)I^{\delta}(h(s))ds\right) > \begin{cases} 1 & \text{if } \delta = 1, \\ 0 & \text{if } \delta \in (0,1), \end{cases} \tag{2.32}$$

then Eq. (EQ) oscillates.

Proof Proceeding similarly to the proof of Theorem 2.1, we again arrive at (2.27). Since h is increasing and an advanced argument, we get $s \leq h(s) \leq h(t)$ for $s \leq t$ and $h(t) \geq t$. Thus, from the fact that ψ is decreasing, we arrive at

$$\psi(h(s)) \geq \psi(h(t)).$$

Now, from the fact $\psi(t)/I(t)$ is nondecrasing, we obtain

$$\begin{aligned}\int_{t_3}^{t} q_2(s)\psi^{\delta}(h(s))ds &\geq \left(\int_{t_3}^{t} q_2(s)ds\right) I^{\delta}(h(t))\frac{\psi^{\delta}(h(t))}{I^{\delta}(h(t))} \\ &\geq \left(\int_{t_3}^{t} q_2(s)ds\right)\left(\frac{I^{\delta}(h(t))}{I^{\delta}(t)}\right)\psi^{\delta}(t). \end{aligned} \tag{2.33}$$

Also, if $s > t$, we have $h(s) \geq h(t)$, and so

$$\frac{\psi^{\delta}(h(s))}{I^{\delta}(h(s))} \geq \frac{\psi^{\delta}(h(t))}{I^{\delta}(h(t))} \geq \frac{\psi^{\delta}(t)}{I^{\delta}(t)}.$$

Thus,

$$\begin{aligned}\int_{t}^{\infty} I(s)q_2(s)\psi^{\delta}(h(s))ds &\geq \int_{t}^{\infty} I(s)q_2(s)I^{\delta}(h(s))\frac{\psi^{\delta}(t)}{I^{\delta}(t)}ds \\ &\geq \left(\frac{1}{I^{\delta}(t)}\int_{t}^{\infty} I(s)q_2(s)I^{\delta}(h(s))ds\right)\psi^{\delta}(t). \end{aligned} \tag{2.34}$$

Using (2.33) and (2.34) in (2.27) yields

$$\psi^{1-\delta}(t) \geq \frac{I^{\delta}(h(t))}{I^{\delta-1}(t)}\int_{t_3}^{t} q_2(s)ds + \frac{1}{I^{\delta}(t)}\int_{t}^{\infty} I(s)q_2(s)I^{\delta}(h(s))ds.$$

The proof's remainder part is similar to that of Theorem 2.1 and is therefore skipped. The proof is complete. □

Theorem 2.5 *Let (1.1) and (2.1) are satisfied and let $\phi_1(t) \geq \vartheta(t)$. If*

$$\begin{aligned}\limsup_{t\to\infty}\Bigg(&I(h(t))\int_{t_0}^{t} q_2(s)ds + I^{1-\delta}(h(t))\int_{t}^{h(t)} I^{\delta}(h(s))q_2(s)ds \\ &+I^{-\delta}(h(t))\int_{h(t)}^{\infty} I(s)q_2(s)I^{\delta}(h(s))ds\Bigg) > \begin{cases} 1 & \text{if } \delta = 1, \\ 0 & \text{if } \delta \in (0,1), \end{cases} \end{aligned} \tag{2.35}$$

then Eq. (EQ) oscillates.

Proof Processing as in the proof of Theorem 2.1 leads again to (2.27), that can be expressed as

$$\psi(h(t)) \geq I(h(t)) \int_{t_3}^{t} q_2(s)\psi^{\delta}(h(s))ds + I(h(t)) \int_{t}^{h(t)} q_2(s)\psi^{\delta}(h(s))ds + \int_{h(t)}^{\infty} I(s)q_2(s)\psi^{\delta}(h(s))ds.$$

This proof's remainder part is easily followed from the fact that ψ is decreasing and $\psi(t)/I(t)$ is increasing as in the proof of Theorem 2.1. □

We conclude the study with the following illustrative exmples and the remark for further research.

Example 2.6 Consider the equation

$$\left(t^2\left(u(t) + 2tu\left(\frac{t}{2}\right)\right)'\right)' + \int_1^2 \left(12t + \frac{12}{\mu}\right) u\left(\frac{t}{4} + \frac{1}{4\mu}\right) d\mu = 0, \quad t \geq 1. \tag{2.36}$$

Here $a(t) = t^2$, $p(t) = 2t$, $\vartheta(t) = t/2$, $\phi(t,\mu) = t/4 + 1/4\mu$, $\delta = 1$, $\alpha = 1$, $\beta = 2$, and $q(t,\mu) = 12t + 12/\mu$. Then

$$\vartheta^{-1}(t) = 2t, \ \vartheta^{-1}(\vartheta^{-1}(t)) = 4t, \ \phi_2(t) = (t+1)/4 \text{ and } h(t) = (t+1)/2,$$

$$I(t) = \frac{1}{t}, \ \pi(t) \geq 7/32t, \ q_1(t) = 12, \text{ and } q_2(t) \geq 21/2.$$

Since $I(t_0) = 1$ and

$$\int_{t_0}^{\infty} I(s)q_1(s)ds = \int_1^{\infty} \frac{12}{s} ds = \infty,$$

conditions (1.1) and (2.1) are satisfied. Also, one can easily see that condition (2.2) is satisfied. Now, all assumptions of Theorems 2.1 are fulfilled. Therefore, by Theorem 2.1, Eq. (2.36) oscillates.

Example 2.7 Consider the equation

$$\left(t^3\left(u(t) + 4u\left(\frac{t}{3}\right)\right)'\right)' + \int_1^2 (t+\mu)\, u^{1/3}\,(2t-\mu)\, d\mu = 0, \quad t \geq 5. \tag{2.37}$$

Here $a(t) = t^3$, $p(t) = 4$, $\vartheta(t) = t/3$, $\phi(t,\mu) = 2t - \mu$, $\delta = 1/3$, $\alpha = 1$, $\beta = 2$, and $q(t,\mu) = t + \mu$. Then

$$\vartheta^{-1}(t) = 3t, \ \vartheta^{-1}(\vartheta^{-1}(t)) = 9t, \ \phi_2(t) = 2t - 1 \text{ and } h(t) = 6t - 3,$$

$$I(t) = \frac{1}{2t^2}, \ \pi(t) = 3/16, \ q_1(t) = (2t+3)/\sqrt[3]{32}, \text{ and } q_2(t) = \sqrt[3]{3/128}(2t+3).$$

Since $I(t_0) = 1/50$ and

$$\int_{t_0}^{\infty} I(s)q_1(s)ds = \int_5^{\infty} \frac{2s+3}{2\sqrt[3]{32}s^2} ds = \infty,$$

conditions (1.1) and (2.1) are satisfied. Also, one can easily see that condition (2.32) is satisfied. Now, all assumptions of Theorems 2.4 are fulfilled. Therefore, by Theorem 2.4, Eq. (2.37) oscillates.

Remark 2.8 The oscillation criteria established in this paper are also valid in the case when the function ϕ in condition (iv) is nondecreasing in its second variable. To set up these results, one should replace

$$\phi_1(t) := \phi(t, \beta) \quad \text{by} \quad \overline{\phi}_1(t) := \phi(t, \alpha)$$

and

$$\phi_2(t) := \phi(t, \alpha) \quad \text{by} \quad \overline{\phi}_2(t) := \phi(t, \beta).$$

Remark 2.9 In the case when $p(t) \leq -1$ with $p(t) \not\equiv -1$ for large t, the problem of obtaining oscillation criteria for all solutions of Eq. (EQ) remains open at the moment.

3 Conclusion

In this study, we obtain several novel oscillation results that can be applied not only to the case when the neutral coefficient function $p(t)$ is unbounded from above, but also to the case when the function $p(t)$ is constant or bounded. The oscillation criteria established in this manuscript are extendable to the differential equations of higher-order, for instance to the equation of the form

$$\left(a(t)\left(\psi^{(m-1)}(t)\right)^{\theta}\right)' + \int_{\alpha}^{\beta} q(t,\mu)u^{\delta}(\phi(t,\mu))d\mu = 0,$$

under either of the conditions

$$\int_{t_0}^{\infty} a^{-1/\theta}(s)ds = \infty$$

or

$$\int_{t_0}^{\infty} a^{-1/\theta}(s)ds < \infty,$$

where θ is the quotient of odd natural numbers, $m \geq 2$ is an even natural number, and all the other functions appearing in the equation are defined as in the present study.

Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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