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# Some new paranormed difference sequence spaces and weighted core

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ABSTRACT

In this study, we define new paranormed sequence spaces by combining a generalized weighted mean and a difference operator. Furthermore, we compute the $\alpha$-,$\beta$- and $\gamma$- duals and obtain bases for these sequence spaces. Besides this, we characterize the matrix transformations from the new paranormed sequence spaces to the spaces $c_0(q)$, $c(q)$, $\ell(q)$ and $\ell_\infty(q)$. Finally, weighted core of a complex-valued sequence has been introduced, and we prove some inclusion theorems related to this new type of core.


## 1. Preliminaries, background and notation

By $\omega$, we shall denote the space of all real valued sequences. Any vector subspace of $\omega$ is called as a *sequence space*. We shall write $\ell_\infty$, $c$ and $c_0$ for the spaces of all bounded, convergent and null sequences, respectively. Also by $bs$, $cs$, $\ell_1$ and $\ell_p$, we denote the spaces of all bounded, convergent, absolutely and $p$-absolutely convergent series, respectively, $1 < p < \infty$.

A linear topological space $X$ over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha$'s in $\mathbb{R}$ and all $x$'s in $X$, where $\theta$ is the zero vector in the linear space $X$.

Assume here and after that $(p_k)$ be a bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear spaces $c(p)$, $c_0(p)$, $\ell_\infty(p)$ and $\ell(p)$ were defined by Maddox [1,2] (see also [3,4]) as follows:

$$c(p) = \left\{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{C}\right\},$$

$$c_0(p) = \left\{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\},$$

$$\ell_\infty(p) = \left\{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\right\}$$

and

$$\ell(p) = \left\{x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty\right\},$$

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which are the complete spaces paranormed by

$$h_1(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/M} \quad \text{iff} \quad \inf_{p_k} > 0 \quad \text{and} \quad h_2(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/M},$$

respectively. We shall assume throughout that $p_k^{-1} + (p'_k)^{-1} = 1$ provided $1 < \inf p_k < H < \infty$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. By $\mathcal{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$, respectively.

For the sequence spaces $X$ and $Y$, define the set $S(X, Y)$ by

$$S(X, Y) = \{z = (z_k) : xz = (x_k z_k) \in Y \text{ for all } x \in X\}. \tag{1}$$

With the notation of (1), the $\alpha$-, $\beta$- and $\gamma$-duals of a sequence space $X$, which are respectively denoted by $X^\alpha$, $X^\beta$ and $X^\gamma$, are defined by

$$X^\alpha = S(X, \ell_1), \qquad X^\beta = S(X, cs) \quad \text{and} \quad X^\gamma = S(X, bs).$$

Let $(X, h)$ be a paranormed space. A sequence $(b_k)$ of the elements of $X$ is called a basis for $X$ if and only if, for each $x \in X$, there exists a unique sequence $(\alpha_k)$ of scalars such that

$$h\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) \to 0 \quad \text{as } n \to \infty.$$

The series $\sum \alpha_k b_k$ which has the sum $x$ is then called the expansion of $x$ with respect to $(b_n)$ and written as $x = \sum \alpha_k b_k$. Let $X, Y$ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then, we say that $A$ defines a matrix mapping from $X$ into $Y$, and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = ((Ax)_n)$, the $A$-transform of $x$, is in $Y$, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{2}$$

By $(X : Y)$, we denote the class of all matrices $A$ such that $A : X \to Y$. Thus, $A \in (X : Y)$ if and only if the series on the right-hand side of (2) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in Y$ for all $x \in X$. A sequence $x$ is said to be $A$-summable to $\alpha$ if $Ax$ converges to $\alpha$ which is called as the $A$-limit of $x$. If $X$ and $Y$ are equipped with the limits $X - \lim$ and $Y - \lim$, respectively, $A \in (X : Y)$ and $Y - \lim_n A_n(x) = X - \lim_k x_k$ for all $x \in X$, then we say that $A$ regularly maps $X$ into $Y$ and write $A \in (X : Y)_{reg}$.

Let $x = (x_k)$ be a sequence in $\mathbb{C}$, the set of all complex numbers, and $R_k$ be the least convex closed region of complex plane containing $x_k, x_{k+1}, x_{k+2}, \ldots$. The Knopp Core (or $\mathcal{K}$-*core*) of $x$ is defined by the intersection of all $R_k$ $(k = 1, 2, \ldots)$, (see [5, pp. 137]). In [6], it is shown that

$$\mathcal{K}\text{-}core(x) = \bigcap_{z\in\mathbb{C}} B_x(z)$$

for any bounded sequence $x$, where $B_x(z) = \left\{w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z|\right\}$.

Let $E$ be a subset of $\mathbb{N}$. The natural density $\delta$ of $E$ is defined by

$$\delta(E) = \lim_n \frac{1}{n} |\{k \leq n : k \in E\}|$$

where $|\{k \leq n : k \in E\}|$ denotes the number of elements of $E$ not exceeding $n$. A sequence $x = (x_k)$ is said to be statistically convergent to a number $l$, if $\delta(\{k : |x_k - l| \geq \varepsilon\}) = 0$ for every $\varepsilon$. In this case we write $st - \lim x = l$, [7]. By $st$ we denote the space of all statistically convergent sequences.

In [8], the notion of the statistical core (or $st$-*core*) of a complex valued sequence has been introduced by Fridy and Orhan and it is shown for a statistically bounded sequence $x$ that

$$st\text{-}core(x) = \bigcap_{z\in\mathbb{C}} C_x(z),$$

where $C_x(z) = \left\{w \in \mathbb{C} : |w - z| \leq st - \limsup_k |x_k - z|\right\}$. The core theorems have been studied by many authors. For instance, see [9–13] and the others.

We write by $\mathcal{U}$ for the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}$. For $u \in \mathcal{U}$, let $1/u = (1/u_n)$. Let $u, v \in \mathcal{U}$ and let us define the matrix $G(u, v) = (g_{nk})$ and $\Delta = (\delta_{nk})$ as follows:

$$g_{nk} = \begin{cases} u_n v_k, & (0 \leq k \leq n), \\ 0, & (k > n), \end{cases} \qquad \delta_{nk} = \begin{cases} (-1)^{n-k}, & (n - 1 \leq k \leq n), \\ 0, & (0 \leq k < n - 1 \text{ or } k > n), \end{cases}$$

for all $n, k \in \mathbb{N}$, where $u_n$ depends only on $n$ and $v_k$ only on $k$. The matrix $G(u, v)$, defined above, is called as generalized weighted mean or factorable matrix.

The main purpose of this study is to introduce the sequence spaces $c_0(u, v; p, \Delta)$, $c(u, v; p, \Delta)$, $\ell_\infty(u, v; p, \Delta)$ and $\ell(u, v; p, \Delta)$ which are the set of all sequences whose $G(u, v; \Delta)$-transforms are in the spaces $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$, respectively, where $G(u, v; \Delta)$ denotes the matrix $G(u, v; \Delta) = G(u, v)\Delta = (g^{\Delta}_{nk})$ defined by

$$g^{\Delta}_{nk} = \begin{cases} u_n(v_k - v_{k+1}), & (0 \le k \le n-1), \\ u_k v_k, & (k = n), \\ 0, & (k > n). \end{cases}$$

Also, we have investigated some topological structures, which have completeness, the $\alpha$-, $\beta$- and $\gamma$-duals, and the bases of these sequence spaces. Besides this, we characterize some matrix mappings on these spaces. Finally, we have defined Weighted Core (*Z-core*) of a sequence and characterized some class of matrices for which $Z\text{-}core(Ax) \subseteq \mathcal{K}\text{-}core(x)$ and $Z\text{-}core(Ax) \subseteq st_A\text{-}core(x)$ for all $x \in \ell_\infty$.

## 2. The paranormed sequence spaces $\lambda(u, v; p, \Delta)$ for $\lambda \in \{c_0(p), c(p), \ell_\infty(p), \ell(p)\}$

In this section, we define the new sequence spaces $\lambda(u, v; p, \Delta)$ for $\lambda \in \{c_0(p), c(p), \ell_\infty(p), \ell(p)\}$ derived by using the generalized weighted mean and difference operator, and prove that these sequence spaces are the complete paranormed linear metric spaces and compute their $\alpha$-, $\beta$- and $\gamma$-duals. Moreover, we give the basis for the spaces $\lambda(u, v; p, \Delta)$ for $\lambda \in \{c_0(p), c(p), \ell(p)\}$.

For a sequence space $X$, the matrix domain $X_A$ of an infinite matrix $A$ is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}. \quad (3)$$

In [14], Choudhary and Mishra have defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that $S$-transforms are in $\ell(p)$, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1, & (0 \le k \le n), \\ 0, & (k > n). \end{cases}$$

Başar and Altay [15] have recently examined the space $bs(p)$ which is formerly defined by Başar in [16] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$. More recently, Altay and Başar have studied the sequence spaces $r^t(p)$, $r^t_\infty(p)$ in [17] and $r^t_c(p)$, $r^t_0(p)$ in [18] which are derived by the Riesz means from the sequence spaces $\ell(p)$, $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ of Maddox, respectively. With the notation of (3), the spaces $\overline{\ell(p)}$, $bs(p)$, $r^t(p)$, $r^t_\infty(p)$, $r^t_c(p)$ and $r^t_0(p)$ may be redefined by

$$\overline{\ell(p)} = [\ell(p)]_S, \qquad bs(p) = [\ell_\infty(p)]_S, \qquad r^t(p) = [\ell(p)]_{R^t},$$
$$r^t_\infty(p) = [\ell_\infty(p)]_{R^t}, \qquad r^t_c(p) = [c(p)]_{R^t}, \qquad r^t_0(p) = [c_0(p)]_{R^t}.$$

Following [14,15,17,18], we define the sequence spaces $\lambda(u, v; p, \Delta)$ for $\lambda \in \{c_0(p), c(p), \ell_\infty(p), \ell(p)\}$ by

$$\lambda(u, v; p, \Delta) = \left\{ x = (x_k) \in \omega : \left( \sum_{k=1}^{n} u_n v_k \Delta x_k \right) \in \lambda \right\}.$$

With notation (3), we may redefine the spaces $c_0(u, v; p, \Delta)$, $c(u, v; p, \Delta)$, $\ell_\infty(u, v; p, \Delta)$ and $\ell(u, v; p, \Delta)$ as follows:

$$c_0(u, v; p, \Delta) = \{c_0(p)\}_{G(u,v;\Delta)}, \qquad c(u, v; p, \Delta) = \{c(p)\}_{G(u,v;\Delta)},$$
$$\ell_\infty(u, v; p, \Delta) = \{\ell_\infty(p)\}_{G(u,v;\Delta)}, \qquad \ell(u, v; p, \Delta) = \{\ell(p)\}_{G(u,v;\Delta)}.$$

This definition includes the following special cases.

(i) If $p = e$, then $\lambda(u, v; p, \Delta) = \lambda(u, v, \Delta)$ (cf. [19]).
(ii) If $v = 1 + r^k$, $u = (1/(n + 1))$, $p = e$, $\lambda = c$ and $\lambda = c_0$, then $\lambda(u, v; p, \Delta) = a^r_0(\Delta)$ and $\lambda(u, v; p, \Delta) = a^r_c(\Delta)$ (cf. [20]).
(iii) If $v = 1 + r^k$, $u = (1/(n + 1))$, $p = e$, $\lambda = \ell_\infty$, then $\lambda(u, v; p, \Delta) = a^r_\infty(\Delta)$ (cf. [21]).
(iv) If $v = 1 + r^k$, $u = (1/(n + 1))$, $p = e$, $\lambda = \ell_p$, then $\lambda(u, v; p, \Delta) = a^r_p(\Delta)$ (cf. [22]).
(v) If $v = q_k$, $u = (1/\sum_{k=0}^{n} q_k)$, $\lambda = \ell_\infty(p)$, $c(p)$ and $\lambda = c_0(p)$, then $\lambda(u, v; p, \Delta) = r^q_\infty(p, \Delta)$, $\lambda(u, v; p, \Delta) = r^q_c(p, \Delta)$ and $\lambda(u, v; p, \Delta) = r^q_0(p, \Delta)$ (cf. [23]).
(vi) If $v = q_k$, $u = (1/\sum_{k=0}^{n} q_k)$ and $\lambda = \ell(p)$, then $\lambda(u, v; p, \Delta) = r^q(p, \Delta)$ (cf. [24]).

Define the sequence $y = (y_k)$, which will be frequently used as the $G(u, v; \Delta)$-transform of a sequence $x = (x_k)$, i.e.

$$y_k = \sum_{j=0}^{k} u_k v_j \Delta x_j = \sum_{j=0}^{k-1} u_k \nabla v_j x_j + u_k v_k x_k; \quad (k \in \mathbb{N}), \quad (4)$$

where $\nabla v_j = (v_j - v_{j+1})$. Since the proof may also be obtained in the similar way as for the other spaces, to avoid the repetition of the similar statements, we give the proof only for one of those spaces. Now, we may begin with the following theorem which is essential in the study.

**Theorem 1.** (i) *The sequence spaces $\lambda(u, v; p, \Delta)$ for $\lambda \in \{c_0(p), c(p), \ell_\infty(p)\}$ are the complete linear metric spaces paranormed by $g$, defined by*

$$g(x) = \sup_{k\in\mathbb{N}} \left| \sum_{j=0}^{k} u_k v_j \Delta x_j \right|^{p_k/M}.$$

*$g$ is a paranorm for the spaces $c(u, v; p, \Delta)$ and $\ell_\infty(u, v; p, \Delta)$ only in the trivial case $\inf p_k > 0$ when $c(u, v; p, \Delta) = c(u, v; \Delta)$ and $\ell_\infty(u, v; p, \Delta) = \ell_\infty(u, v; \Delta)$.*

(ii) *$\ell(u, v; p, \Delta)$ is a complete linear metric space paranormed by*

$$g^*(x) = \left( \sum_k \left| \sum_{j=0}^{k} u_k v_j \Delta x_j \right|^{p_k} \right)^{1/M}.$$

**Proof.** We prove the theorem for the space $c_0(u, v; p, \Delta)$. The linearity of $c_0(u, v; p, \Delta)$ with respect to the coordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $s, x \in c_0(u, v; p, \Delta)$ (see [25, p. 30]):

$$\sup_{k\in\mathbb{N}} \left| \sum_{j=0}^{k} u_k v_j \Delta(s_j + x_j) \right|^{p_k/M} \le \sup_{k\in\mathbb{N}} \left| \sum_{j=0}^{k} u_k v_j \Delta s_j \right|^{p_k/M} + \sup_{k\in\mathbb{N}} \left| \sum_{j=0}^{k} u_k v_j \Delta x_j \right|^{p_k/M} \tag{5}$$

and for any $\alpha \in \mathbb{R}$ (see [2]),

$$|\alpha|^{p_k} \le \max\{1, |\alpha|^M\}. \tag{6}$$

It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in c_0(u, v; p, \Delta)$. Again inequalities (5) and (6) yield the subadditivity of $g$ and

$$g(\alpha x) \le \max\{1, |\alpha|\} g(x).$$

Let $\{x^n\}$ be any sequence of the points $x^n \in c_0(u, v; p, \Delta)$ such that $g(x^n - x) \to 0$ and $(\alpha_n)$ also be any sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality

$$g(x^n) \le g(x) + g(x^n - x)$$

holds by the subadditivity of $g$, $\{g(x^n)\}$ is bounded and we thus have

$$\begin{aligned} g(\alpha_n x^n - \alpha x) &= \sup_{k\in\mathbb{N}} \left| \sum_{j=0}^{k} u_k v_j \Delta(\alpha_n x_j^n - \alpha x_j) \right|^{p_k/M} \\ &\le |\alpha_n - \alpha|\, g(x^n) + |\alpha|\, g(x^n - x), \end{aligned}$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, $g$ is a paranorm on the space $c_0(u, v; p, \Delta)$.

It remains to prove the completeness of the space $c_0(u, v; p, \Delta)$. Let $\{x^i\}$ be any Cauchy sequence in the space $c_0(u, v; p, \Delta)$, where $x^i = \{x_0^{(i)}, x_1^{(i)}, \ldots\}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^i - x^j) < \frac{\varepsilon}{2}$$

for all $i, j \ge n_0(\varepsilon)$. We obtain by using definition of $g$ for each fixed $k \in \mathbb{N}$ that

$$\begin{aligned} |\{G(u, v; \Delta)x^i\}_k - \{G(u, v; \Delta)x^j\}_k|^{p_k/M} &\le \sup_{k\in\mathbb{N}} |\{G(u, v; \Delta)x^i\}_k - \{G(u, v; \Delta)x^j\}_k|^{p_k/M} \\ &< \frac{\varepsilon}{2} \end{aligned} \tag{7}$$

for every $i, j \ge n_0(\varepsilon)$, which leads us to the fact that $\{(G(u, v; \Delta)x^0)_k, (G(u, v; \Delta)x^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say

$$\{G(u, v; \Delta)x^i\}_k \to \{G(u, v; \Delta)x\}_k$$

as $i \to \infty$. Using these infinitely many limits $(G(u, v; \Delta)x)_0, (G(u, v; \Delta)x)_1, \ldots$, we define the sequence $\{(G(u, v; \Delta)x)_0, (G(u, v; \Delta)x)_1, \ldots\}$. We have from (7) with $j \to \infty$ that

$$|\{G(u, v; \Delta)x^i\}_k - \{G(u, v; \Delta)x\}_k|^{p_k/M} \le \frac{\varepsilon}{2} \quad (i \ge n_0(\varepsilon)) \tag{8}$$

for every fixed $k \in \mathbb{N}$. Since $x^i = \{x_k^{(i)}\} \in c_0(u, v; p, \Delta)$,

$$|\{G(u, v; \Delta)x^i\}_k|^{p_k/M} < \frac{\varepsilon}{2}$$

for all $k \in \mathbb{N}$. Therefore, we obtain (8) that

$$\begin{aligned}|\{G(u, v; \Delta)x\}_k|^{p_k/M} &\leq |\{G(u, v; \Delta)x\}_k - \{G(u, v; \Delta)x^i\}_k|^{p_k/M} + |\{G(u, v; \Delta)x^i\}_k|^{p_k/M} \\ &< \varepsilon \quad (i \geq n_0(\varepsilon)).\end{aligned}$$

This shows that the sequence $\{G(u, v; \Delta)x\}$ belongs to the space $c_0(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $c_0(u, v; p, \Delta)$ is complete and this concludes the proof. □

**Theorem 2.** *The sequence spaces $\ell_\infty(u, v; p, \Delta)$, $c(u, v; p, \Delta)$, $c_0(u, v; p, \Delta)$ and $\ell(u, v; p, \Delta)$ are linearly isomorphic to the spaces $\ell_\infty(p)$, $c(p)$, $c_0(p)$ and $\ell(p)$, respectively, where $0 < p_k \leq H < \infty$.*

**Proof.** We establish this for the space $\ell_\infty(u, v; p, \Delta)$. To prove the theorem, we should show the existence of a linear bijection between the spaces $\ell_\infty(u, v; p, \Delta)$ and $\ell_\infty(p)$ for $0 < p_k \leq H < \infty$. With the notation of (4), define the transformations $T$ from $\ell_\infty(u, v; p, \Delta)$ to $\ell_\infty(p)$ by $x \mapsto y = Tx$. The linearity of $T$ is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence $T$ is injective.

Let $y = (y_k) \in \ell_\infty(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k} \frac{1}{v_j}\left[\frac{y_j}{u_j} - \frac{y_{j-1}}{u_{j-1}}\right]; \quad (k \in \mathbb{N}).$$

Then, since

$$(\Delta x)_k = \frac{1}{v_k}\left[\frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}}\right]; \quad (k \in \mathbb{N}),$$

we get that

$$\begin{aligned}g(x) &= \sup_{k\in\mathbb{N}} \left|\sum_{j=0}^{k} u_k v_j \Delta x_j\right|^{p_k/M} \\ &= \sup_{k\in\mathbb{N}} \left|\sum_{j=0}^{k} u_k v_j \frac{1}{v_j}\left[\frac{y_j}{u_j} - \frac{y_{j-1}}{u_{j-1}}\right]\right|^{p_k/M} \\ &= \sup_{k\in\mathbb{N}} |y_k|^{p_k/M} = h_1(y) < \infty.\end{aligned}$$

Thus, we deduce that $x \in \ell_\infty(u, v; p, \Delta)$ and consequently $T$ is surjective and is paranorm preserving. Hence, $T$ is a linear bijection and this says us that the spaces $\ell_\infty(u, v; p, \Delta)$ and $\ell_\infty(p)$ are linearly isomorphic, as desired. □

We shall quote some lemmas which are needed in proving related to the duals our theorems.

**Lemma 1** ([26, *Theorem 5.1.3 with $q_n = 1$*]). *$A \in (\ell_\infty(p) : \ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left|\sum_{k\in K} a_{nk} B^{1/p_k}\right| < \infty \quad \textit{for all integers } B > 1. \tag{9}$$

**Lemma 2** ([26, *Corollary for Theorem 3*]). *Let $p_k > 0$ for every $k$. Then $A \in (\ell_\infty(p) : c)$ if and only if*

$$\sum_k |a_{nk}| B^{1/p_k} \quad \textit{converges uniformly in } n \textit{ for all integers } B > 1, \tag{10}$$

$$\lim_{n\to\infty} a_{nk} = \alpha_k \quad \textit{for all } k. \tag{11}$$

**Lemma 3** ([27, *Theorem 3*]). *Let $p_k > 0$ for every $k$. Then $A \in (\ell_\infty(p) : \ell_\infty)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| B^{1/p_k} < \infty \quad \textit{for all integers } B > 1. \tag{12}$$

**Lemma 4** (*[26, Theorem 5.1.0 with $q_n = 1$]*). *(i) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} a_{nk} B^{-1} \right|^{p'_k} < \infty. \tag{13}$$

*(ii) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in K} a_{nk} \right|^{p_k} < \infty. \tag{14}$$

**Lemma 5** (*[26, Theorem 1(i)–(ii)]*). *(i) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk} B^{-1}|^{p'_k} < \infty. \tag{15}$$

*(ii) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{16}$$

**Lemma 6** (*[26, Corollary for Theorem 1]*). *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : c)$ if and only if (15), (16) hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k, \quad (k \in \mathbb{N}) \tag{17}$$

*also holds.*

**Theorem 3.** *Let $K^* = \{k \in \mathbb{N} : 0 \leq k \leq n\} \cap K$ for $K \in \mathcal{F}$ and $B \in \mathbb{N}_2$. Define the sets $G_1(p), G_2(p), G_3(p), G_4(p), G_5(p), G_6(p), G_7(p)$ and $G_8(p)$ as follows:*

$$G_1(p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K^*} \left[ \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1}{u_j v_k} a_n \right] B^{-1/p_k} \right| < \infty \right\},$$

$$G_2(p) = \left\{ a = (a_k) \in \omega : \sum_n \left| \sum_k \left[ \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1}{u_j v_k} a_n \right] \right| < \infty \right\},$$

$$G_3(p) = \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K^*} \left[ \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1}{u_j v_k} a_n \right] B^{1/p_k} \right| < \infty \right\},$$

$$G_4(p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sum_k \left| \frac{1}{u_k} \left[ \frac{a_k}{v_k} + \left( \frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \sum_{j=k+1}^{n} a_j \right] \right| B^{-1/p_k} < \infty \right\},$$

$$G_5(p) = \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \sum_k \left| \frac{1}{u_k} \left[ \frac{a_k}{v_k} + \left( \frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \sum_{j=k+1}^{n} a_j \right] \right| B^{1/p_k} \text{ converges uniformly in } n \right\},$$

$$G_6(p) = \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \left( \frac{1}{u_k v_k} a_k B^{1/p_k} \right) \in c_0 \right\},$$

$$G_7(p) = \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \sum_k \left| \frac{1}{u_k} \left[ \frac{a_k}{v_k} + \left( \frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \sum_{j=k+1}^{n} a_j \right] \right| B^{1/p_k} < \infty \right\},$$

$$G_8(p) = \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \left\{ \frac{1}{u_k} \left[ \frac{a_k}{v_k} + \left( \frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \sum_{j=k+1}^{n} a_j \right] \right\} B^{1/p_k} \in \ell_\infty \right\}.$$

*Then,*

$$\{c_0(u,v;p,\Delta)\}^\alpha = G_1(p), \quad \{c(u,v;p,\Delta)\}^\alpha = G_1(p) \cap G_2(p), \quad \{\ell_\infty(u,v;p,\Delta)\}^\alpha = G_3(p)$$
$$\{c_0(u,v;p,\Delta)\}^\beta = \{c_0(u,v;p,\Delta)\}^\gamma = G_4(p), \quad \{c(u,v;p,\Delta)\}^\beta = G_4(p) \cap cs,$$
$$\{\ell_\infty(u,v;p,\Delta)\}^\beta = G_5(p) \cap G_6(p),\ \{c(u,v;p,\Delta)\}^\gamma = G_4(p) \cap bs,\ \{\ell_\infty(u,v;p,\Delta)\}^\gamma = G_7(p) \cap G_8(p).$$

**Proof.** We give the proof for the space $\ell_\infty(u, v; p, \Delta)$. Let us take any $a = (a_n) \in \omega$ and define the matrix $D = (d_{nk})$ via the sequence $a = (a_n)$ by

$$d_{nk} = \begin{cases} \dfrac{1}{u_k}\left(\dfrac{1}{v_k} - \dfrac{1}{v_{k+1}}\right), & 0 \le k \le n-1, \\ \dfrac{a_n}{u_n v_n}, & k = n, \\ 0, & k > n, \end{cases}$$

where $n, k \in \mathbb{N}$. Bearing in mind (4) we immediately derive that

$$a_n x_n = \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{1}{u_j v_k} a_n y_j = (Dy)_n; \quad (n \in \mathbb{N}). \tag{18}$$

We therefore observe by (18) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in \ell_\infty(u, v; p, \Delta)$ if and only if $Dy \in \ell_1$ whenever $y \in \ell_\infty(p)$. This means that $a = (a_n) \in \{\ell_\infty(u, v; p, \Delta)\}^\alpha$ whenever $x = (x_n) \in \ell_\infty(u, v; p, \Delta)$ if and only if $D \in (\ell_\infty(p) : \ell_1)$. Then, we derive by Lemma 1 that

$$\{\ell_\infty(u, v; p, \Delta)\}^\alpha = G_3(p).$$

Consider the equation for $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n-1} \frac{1}{u_k}\left[\frac{a_k}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{n} a_j\right] y_k + \frac{1}{u_n v_n} a_n y_n = (Ey)_n \tag{19}$$

where $E = (e_{nk})$ is defined by

$$e_{nk} = \begin{cases} \dfrac{1}{u_k}\left[\dfrac{a_k}{v_k} + \left(\dfrac{1}{v_k} - \dfrac{1}{v_{k+1}}\right)\displaystyle\sum_{j=k+1}^{n} a_j\right], & 0 \le k \le n-1, \\ \dfrac{1}{u_n v_n} a_n, & k = n, \\ 0, & k > n, \end{cases}$$

where $n, k \in \mathbb{N}$. Thus, we deduce from Lemma 2 with (19) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \ell_\infty(u, v; p, \Delta)$ if and only if $Ey \in c$ whenever $y = (y_k) \in \ell_\infty(p)$. This means that $a = (a_n) \in \{\ell_\infty(u, v; p, \Delta)\}^\beta$ whenever $x = (x_n) \in \ell_\infty(u, v; p, \Delta)$ if and only if $E \in (\ell_\infty(p) : c)$. Therefore we derive from (10) and (11) that

$$\sum_k \left| \frac{1}{u_k}\left[\frac{a_k}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{n} a_j\right]\right| B^{1/p_k} \quad \text{converges uniformly in } n$$

and

$$\lim_k \frac{1}{u_k v_k} a_k B^{1/p_k} = 0.$$

This shows that

$$\{\ell_\infty(u, v; p, \Delta)\}^\beta = G_5(p) \cap G_6(p).$$

As this, we deduce from Lemma 3 with (19) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in \ell_\infty(u, v; p, \Delta)$ if and only if $Ey \in \ell_\infty$ whenever $y = (y_k) \in \ell_\infty(p)$. This means that $a = (a_n) \in \{\ell_\infty(u, v; p, \Delta)\}^\gamma$ whenever $x = (x_n) \in \ell_\infty(u, v; p, \Delta)$ if and only if $E \in (\ell_\infty(p) : \ell_\infty)$. Therefore we obtain Lemma 3 that

$$\{\ell_\infty(u, v; p, \Delta)\}^\gamma = G_7(p) \cap G_8(p)$$

and this completes the proof. □

**Theorem 4.** *Let $1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$. Define the sets $G_9(p)$, $G_{10}(p)$ and $G_{11}(p)$ as follows:*

$$G_9(p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K^*} \left[\sum_{j=k-1}^{k} (-1)^{k-j} \frac{1}{u_j v_k} a_n\right] B^{-1} \right|^{p'_k} < \infty \right\},$$

$$G_{10}(p)=\bigcup_{B>1}\left\{a=(a_k)\in\omega:\sum_k\left|\frac{1}{u_k}\left[\frac{a_k}{v_k}+\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{n}a_j\right]B^{-1}\right|^{p'_k}<\infty\right\},$$

$$G_{11}(p)=\bigcup_{B>1}\left\{a=(a_k)\in\omega:\left\{\left(\frac{1}{u_kv_k}a_kB^{-1}\right)^{p'_k}\right\}\in\ell_\infty\right\}.$$

*Then,* $\{\ell(u,v;p,\Delta)\}^\alpha=G_9(p)$, $\{\ell(u,v;p,\Delta)\}^\beta=G_{10}(p)\cap G_{11}(p)\cap cs$, $\{\ell(u,v;p,\Delta)\}^\gamma=G_{10}(p)\cap G_{11}(p)$.

**Proof.** This may be obtained in the similar way, as mentioned in the proof of Theorem 3 with Lemmas 4(i), 5(i), 6 instead of Lemmas 1–3. So, we omit the details. □

**Theorem 5.** *Let* $0<p_k\le 1$ *for every* $k\in\mathbb{N}$. *Define the sets* $G_{12}(p)$, $G_{13}(p)$ *and* $G_{14}(p)$ *as follows:*

$$G_{12}(p)=\left\{a=(a_k)\in\omega:\sup_{K\in\mathcal{F}}\sup_{k\in K}\left|\sum_{n\in K}\left[\sum_{j=k-1}^{k}(-1)^{k-j}\frac{1}{u_jv_k}a_n\right]\right|^{p_k}<\infty\right\},$$

$$G_{13}(p)=\left\{a=(a_k)\in\omega:\sup_{n,k\in\mathbb{N}}\left|\frac{1}{u_k}\left[\frac{a_k}{v_k}+\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{n}a_j\right]\right|^{p_k}<\infty\right\},$$

$$G_{14}(p)=\left\{a=(a_k)\in\omega:\sup_{k\in\mathbb{N}}\left|\frac{1}{u_kv_k}a_k\right|^{p_k}<\infty\right\}.$$

*Then,* $\{\ell(u,v;p,\Delta)\}^\alpha=G_{12}(p)$, $\{\ell(u,v;p,\Delta)\}^\beta=G_{13}(p)\cap G_{14}(p)\cap cs$, $\{\ell(u,v;p,\Delta)\}^\gamma=G_{13}(p)\cap G_{14}(p)$.

**Proof.** This may be obtained in the similar way, as mentioned in the proof of Theorem 3 with Lemmas 4(ii), 5(ii), 6 instead of Lemmas 1–3. So, we omit the details. □

Now, we may give the sequence of the points of the spaces $c_0(u,v;p,\Delta)$, $\ell(u,v;p,\Delta)$ and $c(u,v;p,\Delta)$ which forms a Schauder basis for those spaces. Because of the isomorphism $T$, defined in the proof of Theorem 2, between the sequence spaces $c_0(u,v;p,\Delta)$ and $c_0(p)$, $\ell(u,v;p,\Delta)$ and $\ell(p)$, $c(u,v;p,\Delta)$ and $c(p)$ is onto, the inverse image of the basis of the spaces $c_0(p)$, $\ell(p)$ and $c(p)$ is the basis for our new spaces $c_0(u,v;p,\Delta)$, $\ell(u,v;p,\Delta)$ and $c(u,v;p,\Delta)$, respectively. Therefore, we have the following theorem.

**Theorem 6.** *Let* $\mu_k=(G(u,v;\Delta)x)_k$ *for all* $k\in\mathbb{N}$. *We define the sequence* $b^{(k)}=\{b_n^{(k)}\}_{n\in\mathbb{N}}$ *for every fixed* $k\in\mathbb{N}$ *by*

$$b_n^{(k)}=\begin{cases}\dfrac{1}{u_k}\left[\dfrac{1}{v_k}-\dfrac{1}{v_{k+1}}\right], & 0\le k\le n-1,\\ \dfrac{1}{u_nv_n}, & k=n,\\ 0, & k>n.\end{cases}$$

*Then,*

(a) *The sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $c_0(u,v;p,\Delta)$ *and any* $x\in c_0(u,v;p,\Delta)$ *has a unique representation in the form*

$$x=\sum_k\mu_kb^{(k)}.$$

(b) *The sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $\ell(u,v;p,\Delta)$ *and any* $x\in\ell(u,v;p,\Delta)$ *has a unique representation in the form*

$$x=\sum_k\mu_kb^{(k)}.$$

(c) *The set* $\{z,b^{(k)}\}$ *is a basis for the space* $c(u,v;p,\Delta)$ *and any* $x\in c(u,v;p,\Delta)$ *has a unique representation in the form*

$$x=lz+\sum_k(\mu_k-l)b^{(k)}$$

*where* $l=\lim_{k\to\infty}(G(u,v;\Delta)x)_k$ *and* $z=(z_k)$ *with*

$$z_k=\sum_{j=0}^{k}\frac{1}{v_j}\left[\frac{1}{u_j}-\frac{1}{u_{j-1}}\right].$$

## 3. Some matrix mappings on the sequence spaces $\lambda(u, v; p, \Delta)$

In this section, we characterize the matrix mappings from the sequence spaces $\lambda(u, v; p, \Delta)$ into any given sequence space. We shall write throughout for brevity that

$$\tilde{a}_{nk} = \frac{1}{u_k}\left[\frac{a_{nk}}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{n} a_{nj}\right]$$

for all $n, k \in \mathbb{N}$. We will also use the similar notation with other letters and we the convention that any term with negative subscript is equal to naught.

Suppose throughout that the terms of the infinite matrices $A = (a_{nk})$ and $C = (c_{nk})$ are connected with the relation

$$c_{nk} = \tilde{a}_{nk} \quad \left(\text{or equivalently } a_{nk} = \sum_{j=k}^{\infty} u_k v_k c_{nj}\right) \quad (n, k \in \mathbb{N}). \tag{20}$$

Now, we may give our basic theorem.

**Theorem 7.** *Let $\mu$ be any given sequence space. Then, $A \in (c_0(u, v; p, \Delta) : \mu)$ if and only if $C \in (c_0(p) : \mu)$ and*

$$\left\{\frac{1}{u_k v_k} a_{nk} B^{1/p_k}\right\}_{k\in\mathbb{N}} \in c_0, \quad (\forall n \in \mathbb{N},\ \exists B \in \mathbb{N}_2). \tag{21}$$

**Proof.** Suppose that (21) holds and $\mu$ be any given sequence space. Let $A \in (c_0(u, v; p, \Delta) : \mu)$ and take any $y \in c_0(p)$. Then, $(a_{nk})_{k\in\mathbb{N}} \in [c_0(u, v; p, \Delta)]^\beta$ which yields that (21) is necessary and $(c_{nk})_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, $Cy$ exists and thus letting $m \to \infty$ in the equality

$$\sum_{k=0}^{m} c_{nk} y_k = \sum_{k=0}^{m}\sum_{j=k}^{m} u_k v_k c_{nj} x_k, \quad (n, m \in \mathbb{N})$$

we have that $Cy = Ax$ which leads us to the consequence $C \in (c_0(p) : \mu)$.

Conversely, let $C \in (c_0(p) : \mu)$ and (21) holds, and take any $x \in c_0(u, v; p, \Delta)$. Then, we have $(a_{nk})_{k\in\mathbb{N}} \in [c_0(u, v; p, \Delta)]^\beta$ for each $n \in \mathbb{N}$. Hence, $Ax$ exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m-1} c_{nk} y_k + \frac{1}{u_m v_m} a_{nm} y_m; \quad (n, m \in \mathbb{N})$$

as $m \to \infty$ that $Ax = Cy$ and this shows that $A \in (c_0(u, v; p, \Delta) : \mu)$. This completes the proof. □

**Theorem 8.** *Let $\mu$ be any given sequence space. Then,*

(i) *$A \in (c(u, v; p, \Delta) : \mu)$ if and only if $C \in (c(p) : \mu)$ and (21) holds.*
(ii) *$A \in (\ell_\infty(u, v; p, \Delta) : \mu)$ if and only if $C \in (\ell_\infty(p) : \mu)$ and (21) holds.*

**Proof.** This may be obtained by proceedings as in Theorem 7. So, we omit the details. □

Now, we may quote our corollaries on the characterization of some matrix classes concerning with the sequence spaces $c_0(u, v; p, \Delta)$, $c(u, v; p, \Delta)$ and $\ell_\infty(u, v; p, \Delta)$. Prior to giving the corollaries, let us suppose that $(q_n)$ is a non-decreasing bounded sequence of positive real numbers and consider the following conditions:

$$\sup_{n\in\mathbb{N}}\left[\sum_k |\tilde{a}_{nk}| B^{1/p_k}\right]^{q_n} < \infty, \quad (\forall B \in \mathbb{N}), \tag{22}$$

$$\sup_{K\in\mathcal{F}}\sum_n \left|\sum_{k\in K} \tilde{a}_{nk} B^{1/p_k}\right|^{q_n} < \infty, \quad (\forall B \in \mathbb{N}), \tag{23}$$

$$\sup_{n\in\mathbb{N}}\sum_k \left|\tilde{a}_{nk}\right| B^{1/p_k} < \infty, \quad (\forall B \in \mathbb{N}), \tag{24}$$

$$\exists(\alpha_k) \subset \mathbb{R} \ni \lim_{n\to\infty}\left[\sum_k \left|\tilde{a}_{nk} - \alpha_k\right| B^{1/p_k}\right]^{q_n} = 0, \quad (\forall B \in \mathbb{N}), \tag{25}$$

$$\sup_{n\in\mathbb{N}}\left[\sum_k \left|\tilde{a}_{nk}\right| B^{-1/p_k}\right]^{q_n} < \infty, \quad (\exists B \in \mathbb{N}), \tag{26}$$

$$\sup_{n\in\mathbb{N}}\left|\sum_k \tilde{a}_{nk}\right|^{q_n}<\infty, \tag{27}$$

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}\tilde{a}_{nk}B^{-1/p_k}\right|^{q_n}<\infty, \quad (\exists B\in\mathbb{N}), \tag{28}$$

$$\sum_n\left|\sum_k \tilde{a}_{nk}\right|^{q_n}<\infty, \tag{29}$$

$$\exists\alpha\in\mathbb{R}\ni\lim_{n\to\infty}\left|\sum_k \tilde{a}_{nk}-\alpha\right|^{q_n}=0, \tag{30}$$

$$\exists(\alpha_k)\subset\mathbb{R}\ni\lim_{n\to\infty}\left|\tilde{a}_{nk}-\alpha_k\right|^{q_n}=0, \quad (\forall k\in\mathbb{N}), \tag{31}$$

$$\exists(\alpha_k)\subset\mathbb{R}\ni\sup_{n\in\mathbb{N}}K^{1/q_n}\sum_k\left|\tilde{a}_{nk}-\alpha_k\right|B^{-1/p_k}<\infty, \quad (\forall K,\exists B\in\mathbb{N}). \tag{32}$$

**Corollary 1.**

(i) *$A\in(\ell_\infty(u,v;p,\Delta):\ell_\infty(q))$ if and only if (21) and (22) hold.*
(ii) *$A\in(\ell_\infty(u,v;p,\Delta):\ell(q))$ if and only if (21) and (23) hold.*
(iii) *$A\in(\ell_\infty(u,v;p,\Delta):c(q))$ if and only if (21), (24) and (25) hold.*
(iv) *$A\in(\ell_\infty(u,v;p,\Delta):c_0(q))$ if and only if (21) holds and (25) also holds with $\alpha_k=0$ for all $k\in\mathbb{N}$.*

**Corollary 2.**

(i) *$A\in(c(u,v;p,\Delta):\ell_\infty(q))$ if and only if (21), (26) and (27) hold.*
(ii) *$A\in(c(u,v;p,\Delta):\ell(q))$ if and only if (21), (28) and (29) hold.*
(iii) *$A\in(c(u,v;p,\Delta):c(q))$ if and only if (21), (30), (31) and (32) hold, and (26) also holds with $q_n=1$ for all $n\in\mathbb{N}$.*
(iv) *$A\in(c(u,v;p,\Delta):c_0(q))$ if and only if (21) holds, and (30)–(32) also hold with $\alpha=0$, $\alpha_k=0$ for all $k\in\mathbb{N}$, respectively.*

**Corollary 3.**

(i) *$A\in(c_0(u,v;p,\Delta):\ell_\infty(q))$ if and only if (21) and (26) hold.*
(ii) *$A\in(c_0(u,v;p,\Delta):\ell(q))$ if and only if (21) and (28) hold.*
(iii) *$A\in(c_0(u,v;p,\Delta):c(q))$ if and only if (21), (31) and (32) hold and (26) also holds with $q_n=1$ for all $n\in\mathbb{N}$.*
(iv) *$A\in(c_0(u,v;p,\Delta):c_0(q))$ if and only if (21) holds and (25) also holds with $\alpha_k=0$ for all $k\in\mathbb{N}$.*

Finally, we shall present Theorems 9 and 10 that characterize some matrix mappings on the sequence space $\ell(u,v;p,\Delta)$ without proofs.

**Theorem 9.** (i) *Let $1<p_k\le H<\infty$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(u,v;p,\Delta):\ell_\infty)$ if and only if there exists an integer $B>1$ such that*

$$\sup_{n\in\mathbb{N}}\sum_k|\tilde{a}_{nk}B^{-1}|^{p'_k}<\infty, \tag{33}$$

$$\left\{\left[\frac{1}{u_kv_k}a_{nk}B^{-1}\right]^{p'_k}\right\}_{k\in\mathbb{N}}\in\ell_\infty, \quad (n\in\mathbb{N}). \tag{34}$$

(ii) *Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(u,v;p,\Delta):\ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}}|\tilde{a}_{nk}|^{p_k}<\infty, \tag{35}$$

$$\left\{\left[\frac{1}{u_kv_k}a_{nk}\right]^{p_k}\right\}_{k\in\mathbb{N}}\in\ell_\infty, \quad (n\in\mathbb{N}). \tag{36}$$

**Theorem 10.** *Let $0<p_k\le H<\infty$ for all $k\in\mathbb{N}$. Then, $A\in(\ell(u,v;p,\Delta):c)$ if and only if (33)–(36) hold, and there is a sequence $(\alpha_k)$ of the scalars such that*

$$\lim_{n\to\infty}\tilde{a}_{nk}=\alpha_k; \quad (k\in\mathbb{N}). \tag{37}$$

If the sequence space $c$ is replaced by the space $c_0$, then Theorem 10 is reduced.

**Corollary 4.** *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(u, v; p, \Delta) : c_0)$ if and only if (33)–(36) hold, and (37) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

## 4. Weighted core

Using the converge domain of the matrix $G(u, v)$, the new sequence spaces $Z(u, v, c)$ and $Z(u, v, c_0)$ have been constructed in [28] and their some properties have been investigated. In the present section, we first introduce a new type core, $Z$-*core*, of a complex valued sequence and also determine the necessary and sufficient conditions on a matrix $B$ for which $Z\text{-}core(Bx) \subseteq \mathcal{K}\text{-}core(x)$ and $Z\text{-}core(Bx) \subseteq st_A\text{-}core(x)$ for all $x \in \ell_\infty$.

To do these, we need to characterize the classes $(c : Z(u, v, c))_{reg}$ and $(st(A) \cap \ell_\infty : Z(u, v, c))_{reg}$. For brevity, in what follows we write $\tilde{b}_{nk}$ in place of

$$u_n \sum_{k=0}^{n} v_k b_{nk}; \quad (n, k \in \mathbb{N}).$$

**Lemma 7.** *$B \in (\ell_\infty : Z(u, v, c))$ if and only if*

$$\|B\| = \sup_n \sum_k |\tilde{b}_{nk}| < \infty, \tag{38}$$

$$\lim_n \tilde{b}_{nk} = \alpha_k \quad \text{for each } k, \tag{39}$$

$$\lim_n \sum_k |\tilde{b}_{nk} - \alpha_k| = 0. \tag{40}$$

**Proof.** Let $x \in \ell_\infty$ and consider the equality

$$u_n \sum_{j=0}^{n} v_j \sum_{k=0}^{m} b_{jk} x_k = \sum_{k=0}^{m} u_n \sum_{j=0}^{n} v_j b_{jk} x_k; \quad (m, n \in \mathbb{N})$$

which yields as $m \to \infty$ that

$$u_n \sum_{j=0}^{n} v_j (Bx)_j = (Dx)_n; \quad (n \in \mathbb{N}), \tag{41}$$

where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} u_n \sum_{j=0}^{n} v_j b_{jk} & 0 \leq k \leq n, \\ 0 & k > n. \end{cases}$$

Therefore, one can easily see that $B \in (\ell_\infty : Z(u, v, c))$ if and only if $D \in (\ell_\infty : c)$ (see [29]) and this completes the proof. □

**Lemma 8.** *$B \in (c : Z(u, v, c))_{reg}$ if and only if (38) and (39) of the Lemma 7 hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$ and*

$$\lim_n \sum_k \tilde{b}_{nk} = 1. \tag{42}$$

**Proof.** Since the proof is easy we omit it. □

**Lemma 9.** *$B \in (st(A) \cap \ell_\infty : Z(u, v, c))_{reg}$ if and only if $B \in (c : Z(u, v, c))_{reg}$ and*

$$\lim_n \sum_{k \in E} |\tilde{b}_{nk}| = 0 \tag{43}$$

*for every $E \subset \mathbb{N}$ with $\delta_A(E) = 0$.*

**Proof.** Because of $c \subset st \cap \ell_\infty$, $B \in (c : Z(u, v, c))_{reg}$. Now, for any $x \in \ell_\infty$ and a set $E \subset \mathbb{N}$ with $\delta(E) = 0$, let us define the sequence $z = (z_k)$ by

$$z_k = \begin{cases} x_k, & k \in E \\ 0, & k \notin E. \end{cases}$$

Then, since $z \in st_0, Az \in Z(u, v, c_0)$, where $Z(u, v, c_0)$ is the space of sequences which the $G(u, v)$- transforms of them in $c_0$. Also, since

$$\sum_k \tilde{b}_{nk} z_k = \sum_{k \in E} \tilde{b}_{nk} x_k,$$

the matrix $D = (d_{nk})$ defined by $d_{nk} = \tilde{b}_{nk}$ $(k \in E)$ and $d_{nk} = 0$ $(k \notin E)$ is in the class $(\ell_\infty : Z(u, v, c))$. Hence, the necessity of (43) follows from Lemma 7.

Conversely, let $x \in st(A) \cap \ell_\infty$ with $st_A - \lim x = l$. Then, the set $E$ defined by $E = \{k : |x_k - l| \geq \varepsilon\}$ has density zero and $|x_k - l| \leq \varepsilon$ if $k \notin E$. Now, we can write

$$\sum_k \tilde{b}_{nk} x_k = \sum_k \tilde{b}_{nk}(x_k - l) + l \sum_k \tilde{b}_{nk}. \tag{44}$$

Since

$$\left| \sum_k \tilde{b}_{nk}(x_k - l) \right| \leq \|x\| \sum_{k \in E} |\tilde{b}_{nk}| + \varepsilon \cdot \|B\|,$$

letting $n \to \infty$ in (44) and using (42) with (43), we have

$$\lim_n \sum_k \tilde{b}_{nk} x_k = l.$$

This implies that $B \in (st(A) \cap \ell_\infty : Z(u, v, c))_{reg}$ and the proof is completed. □

Let us write

$$g_n(x) = u_n \sum_{k=0}^{n} v_k x_k.$$

Then, we can define *Z-core* of a complex sequence as follows.

**Definition 1.** Let $H_n$ be the least closed convex hull containing $g_n(x), g_{n+1}(x), g_{n+2}(x), \ldots$. Then, *Z-core* of $x$ is the intersection of all $H_n$, i.e.,

$$Z\text{-}core(x) = \bigcap_{n=1}^{\infty} H_n.$$

This definition includes the following special cases.

(i) If $v_k = 1 + r^k$ and $u_n = 1/(n+1)$, then $Z$-*core* $= K_r$-*core* (cf. [30]).
(ii) If $v_k = q_k$ and $u_n = 1/\sum_{k=0}^{n} q_k$, then $Z$-*core* $= K_q$-*core* (cf. [31]).

Note that, actually, we define *Z-core* of $x$ by the $\mathcal{K}$-*core* of the sequence $(g_n(x))$. Hence, we can construct the following theorem which is analogue of $\mathcal{K}$-*core*, (see [6]).

**Theorem 11.** *For any $z \in \mathbb{C}$, let*

$$G_x(z) = \left\{ \omega \in \mathbb{C} : |\omega - z| \leq \limsup_n |g_n(x) - z| \right\}.$$

*Then, for any $x \in \ell_\infty$,*

$$Z\text{-}core(x) = \bigcap_{z \in \mathbb{C}} G_x(z).$$

Now, we may give some inclusion theorems. First, we need a lemma.

**Lemma 10** (*[32, Corollary 12]*). *Let $A = (a_{nk})$ be a matrix satisfying $\sum_k |a_{nk}| < \infty$ and $\lim_n a_{nk} = 0$. Then, there exists an $y \in \ell_\infty$ with $\|y\| \leq 1$ such that*

$$\limsup_n \sum_k a_{nk} y_k = \limsup_n \sum_k |a_{nk}|.$$

**Theorem 12.** *Let $B \in (c : Z(u, v, c))_{reg}$. Then, $Z$-core$(Bx) \subseteq \mathcal{K}$-core$(x)$ for all $x \in \ell_\infty$ if and only if*

$$\lim_n \sum_k |\tilde{b}_{nk}| = 1. \tag{45}$$

**Proof.** Since $B \in (c : Z(u, v, c))_{reg}$, the matrix $\tilde{B} = (\tilde{b}_{nk})$ satisfies the conditions of Lemma 10. So, there exists a $y \in \ell_\infty$ with $\|y\| \le 1$ such that

$$\left\{\omega \in \mathbb{C} : |\omega| \le \limsup_n \sum_k \tilde{b}_{nk} y_k\right\} = \left\{\omega \in \mathbb{C} : |\omega| \le \limsup_n \sum_k |\tilde{b}_{nk}|\right\}.$$

On the other hand, since $\mathcal{K}\text{-}core(y) \subseteq B_1(0)$, by the hypothesis

$$\left\{\omega \in \mathbb{C} : |\omega| \le \limsup_n \sum_k |\tilde{b}_{nk}|\right\} \subseteq B_1(0) = \{\omega \in \mathbb{C} : |\omega| \le 1\}$$

which implies (45).

Conversely, let $\omega \in Z\text{-}core(Bx)$. Then, for any given $z \in \mathbb{C}$, we can write

$$\begin{aligned}
|\omega - z| &\le \limsup_n |g_n(Bx) - z| \\
&= \limsup_n \left| z - \sum_k \tilde{b}_{nk} x_k \right| \\
&\le \limsup_n \left| \sum_k \tilde{b}_{nk}(z - x_k) \right| + \limsup_n |z| \left| 1 - \sum_k \tilde{b}_{nk} \right| \\
&= \limsup_n \left| \sum_k \tilde{b}_{nk}(z - x_k) \right|. \qquad (46)
\end{aligned}$$

Now, let $\limsup_k |x_k - z| = l$. Then, for any $\varepsilon > 0$, $|x_k - z| \le l + \varepsilon$ whenever $k \ge k_0$. Hence, one can write that

$$\begin{aligned}
\sum_k \tilde{b}_{nk}(z - x_k) &= \left| \sum_{k<k_0} \tilde{b}_{nk}(z - x_k) + \sum_{k \ge k_0} \tilde{b}_{nk}(z - x_k) \right| \\
&\le \sup_k |z - x_k| \sum_{k<k_0} |\tilde{b}_{nk}| + (l + \varepsilon) \sum_{k \ge k_0} |\tilde{b}_{nk}| \\
&\le \sup_k |z - x_k| \sum_{k<k_0} |\tilde{b}_{nk}| + (l + \varepsilon) \sum_k |\tilde{b}_{nk}|. \qquad (47)
\end{aligned}$$

Therefore, applying $\limsup_n$ under the light of the hypothesis and combining (46) with (47), we have

$$|\omega - z| \le \limsup_n \left| \sum_k \tilde{b}_{nk}(z - x_k) \right| \le l + \varepsilon$$

which means that $\omega \in \mathcal{K}\text{-}core(x)$. This completes the proof. □

**Theorem 13.** *Let $B \in (st(A) \cap \ell_\infty : Z(u, v, c))_{reg}$. Then, $Z\text{-}core(Bx) \subseteq st_A\text{-}core(x)$ for all $x \in \ell_\infty$ if and only if (45) holds.*

**Proof.** Since $st_A\text{-}core(x) \subseteq \mathcal{K}\text{-}core(x)$ for any sequence $x$ [33], the necessity of condition (45) follows from Theorem 12.

Conversely, take $\omega \in Z\text{-}core(Bx)$. Then, we can write again (46). Now, if $st_A - \limsup |x_k - z| = s$, then for any $\varepsilon > 0$, the set $E$ defined by $E = \{k : |x_k - z| > s + \varepsilon\}$ has density zero, (see [33]). Now, we can write

$$\begin{aligned}
\sum_k \tilde{b}_{nk}(z - x_k) &= \left| \sum_{k \in E} \tilde{b}_{nk}(z - x_k) + \sum_{k \notin E} \tilde{b}_{nk}(z - x_k) \right| \\
&\le \sup_k |z - x_k| \sum_{k \in E} |\tilde{b}_{nk}| + (s + \varepsilon) \sum_{k \notin E} |\tilde{b}_{nk}| \\
&\le \sup_k |z - x_k| \sum_{k \in E} |\tilde{b}_{nk}| + (s + \varepsilon) \sum_k |\tilde{b}_{nk}|.
\end{aligned}$$

Thus, applying the operator $\limsup_n$ and using condition (45) with (43), we get that

$$\limsup_n \left| \sum_k \tilde{b}_{nk}(z - x_k) \right| \le s + \varepsilon. \qquad (48)$$

Finally, combining (46) with (48), we have

$$|\omega - z| \leq st_A - \limsup_k |x_k - z|$$

which means that $\omega \in st_A$-$core(x)$ and the proof is completed. □

As a consequence of Theorem 13, we have the following corollary.

**Corollary 5.** *Let $B \in (st \cap \ell_\infty : Z(u, v, c))_{reg}$. Then, $Z$-core$(Bx) \subseteq st$-core$(x)$ for all $x \in \ell_\infty$ if and only if (45) holds.*

## 5. Conclusion

Recently there has been a lot of interest in investigating geometric properties of sequence spaces besides topological and some other usual properties. In the literature, there are many papers concerning the geometric properties of different sequence spaces. For example, see [34–36]. So, from now on some geometric properties of modular sequence space $\ell_\sigma(u, v; p, \Delta)$ such as Kadec–Klee property and uniform Opial property will be examined in the next paper.

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