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Fixed points of non-Newtonian contraction mappings on non-Newtonian metric spaces

Demet Binbaşıoğlu, Serkan Demiriz and Duran Türkoğlu

Abstract. The study of non-Newtonian calculi was started in 1972 by Grossman and Katz. These calculi provide an alternative to the classical calculus and they include the geometric, anageometric and bigeometric calculi, etc. Recently, Çakmak and Başar (2002) have studied the concept of non-Newtonian metric. Also they have given the triangle and Minkowski's inequalities in the sense of non-Newtonian calculus. In this paper, we introduce a fixed point theory by defining some topological structures of the relevant non-Newtonian metric space.

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1. Preliminaries, background and notation

There exists a considerable literature of fixed point theory dealing with results on fixed or common fixed points in different spaces. In the last years, fixed point theorems have been applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other branches of mathematics. Fixed point theory and self-mappings satisfying certain contraction conditions have many applications and have been an important area of various research activities [3, 5, 7].

Recently, multiplicative calculi have been studied by Bashirov, Kurpınar and Özyapıcı [1]. They presented results and applications concerning the well-known properties of derivatives and integrals in the classical calculus. Lately, multiplicative calculi have been extended to the complex-valued functions by Uzer [8] who studied the statements of some principal theorems and concepts of multiplicative complex calculi and showed some similarities between the multiplicative complex calculi and the classical calculus. Also, Özavşar and Çevikel [6] introduced the concept of multiplicative contraction mappings and proved some fixed point theorems of such mappings on a complete multiplicative metric space.

The non-Newtonian calculi are alternatives to the classical calculus of Newton and Leibnitz. They provide a wide variety of mathematical tools for use in science, engineering and mathematics.

The non-Newtonian calculus has many applications in different areas including fractal geometry, image analysis (e.g., in biomedicine), growth/decay analysis (e.g., in economic growth, bacterial growth and radioactive decay), finance (e.g., rates of return), the theory of elasticity in economics, marketing, the economics of climate change, atmospheric temperature, signal processing (electrical engineering), wave theory in physics, quantum physics and gauge theory, information technology, pathogen counts in treated water, actuarial science, tumor therapy and cancer-chemotherapy in medicine, materials science/engineering, demographics, differential equations (including a multiplicative Lorenz system and Runge–Kutta methods), calculus of variations, finite-difference methods, averages of functions, means of two positive numbers, weighted calculus, meta-calculus, approximation theory, least-squares methods, multivariable calculus, complex analysis, functional analysis, probability theory, utility theory, Bayesian analysis, stochastics, decision making, dynamical systems, chaos theory, and dimensional spaces.

In this paper, we consider the non-Newtonian calculus and the non-Newtonian topological structures and we define some topological properties related to non-Newtonian metric spaces. The main purpose of our paper is to introduce a fixed point theory by defining some topological structures of the relevant non-Newtonian metric space. To do this, we need some preparatory knowledge about non-Newtonian calculi. First, we define the non-Newtonian real field and we give the relevant properties due to Çakmak and Başar [2].

A *generator* is defined as an injective function with domain $\mathbb{R}$ and the range of a generator is a subset of $\mathbb{R}$. Each generator generates one arithmetic if and only if each arithmetic is generated by one generator.

Let $\beta = \exp$ be a function defined as

$$\begin{gathered} \beta : \mathbb{R} \to \mathbb{R}^+, \\ x \mapsto \beta(x) = e^x = y, \end{gathered}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Suppose that this function β is a generator, that is, if $\beta = I$, $I(x) = x$ for all $x \in \mathbb{R}$, then β generates the classical arithmetic. If $\beta = \exp$, then β generates geometrical arithmetics.

Define the set $\mathbb{R}(N)$ as

$$\mathbb{R}(N) := \{\beta(x) : x \in \mathbb{R}\},$$

where $\mathbb{R}(N)$ is the set of non-Newtonian real numbers.

All concepts of β-arithmetic have similar properties in classical arithmetic. β-zero, β-one and all β-integers are formed as

$$\ldots, \beta(-1), \beta(0), \beta(1), \ldots.$$

Take any generator β with range A. Then define the operations β-addition, β-substraction, β-multiplication, β-division and β-order in the following way for $x, y \in \mathbb{R}$, respectively:

$$\begin{array}{ll}
\beta\text{-addition} & x \dot{+} y = \beta\{\beta^{-1}(x) + \beta^{-1}(y)\}, \\
\beta\text{-substraction} & x \dot{-} y = \beta\{\beta^{-1}(x) - \beta^{-1}(y)\}, \\
\beta\text{-multiplication} & x \dot{\times} y = \beta\{\beta^{-1}(x) \times \beta^{-1}(y)\}, \\
\beta\text{-division} & x \dot{/} y = \beta\{\beta^{-1}(x) \div \beta^{-1}(y)\}, \\
\beta\text{-order} & x \dot{<} y \Longleftrightarrow \beta(x) < \beta(y).
\end{array}$$

Proposition 1.1 (See [2]**).** $(\mathbb{R}(N), \dot{+}, \dot{\times})$ *is a complete field.*

For $x \in A \subset \mathbb{R}(N)$, a number β-square is described by $x \dot{\times} x$ and denoted by x^{2_N}. The symbol $\sqrt{x}^N$ denotes

$$t = \beta\left\{\sqrt{\beta^{-1}(x)}\right\}$$

which is the unique β nonnegative number whose β-square is equal to x and which means $t^{2_N} = x$, for each β nonnegative number t.

Throughout this paper, x^{p_N} denotes the pth non-Newtonian exponent. Thus we have

$$\begin{aligned}
x^{2_N} &= x \dot{\times} x = \beta\{\beta^{-1}(x) \times \beta^{-1}(x)\} = \beta\left\{\left[\beta^{-1}(x)\right]^2\right\}, \\
x^{3_N} &= x^{2_N} \dot{\times} x = \beta\left\{\beta^{-1}\{\beta[\beta^{-1}(x) \times \beta^{-1}(x)]\} \times \beta^{-1}(x)\right\} \\
&= \beta\left\{\left[\beta^{-1}(x)\right]^3\right\}, \\
&\vdots \\
x^{p_N} &= x^{(p-1)_N} \dot{\times} x = \beta\left\{\left[\beta^{-1}(x)\right]^p\right\}, \\
&\vdots
\end{aligned}$$

We denote by $|x|_N$ the β-absolute value of a number $x \in A \subset \mathbb{R}(N)$ defined as $\beta(|\beta^{-1}(x)|)$ and also

$$\sqrt{x^{2_N}}^N = |x|_N = \beta\{|\beta^{-1}(x)|\}.$$

Thus,

$$|x|_N = \beta\{|\beta^{-1}(x)|\} = \begin{cases} x, & x \dot{>} \beta(0), \\ \beta(0), & x = \beta(0), \\ \beta(0) \dot{-} x, & x \dot{<} \beta(0). \end{cases}$$

For $x_1, x_2 \in A \subseteq \mathbb{R}(N)$, the non-Newtonian distance $|\cdot|_N$ is defined as

$$|x_1 \dot{-} x_2|_N = \beta\{|\beta^{-1}(x_1) - \beta^{-1}(x_2)|\}.$$

This distance is commutative; i.e., $|x_1 \dot{-} x_2|_N = |x_2 \dot{-} x_1|_N$.

Take any $z \in \mathbb{R}(N)$, if $z \dot{>} \beta(0)$, then z is called a *positive non-Newtonian real number*; if $z \dot{<} \beta(0)$, then z is called a *non-Newtonian negative real number* and if $z = \beta(0)$, then z is called an *unsigned non-Newtonian real number*. Non-Newtonian positive real numbers are denoted by $\mathbb{R}^+(N)$ and non-Newtonian negative real numbers by $\mathbb{R}^-(N)$ (see [4]).

The fundamental properties provided in the classical calculus is provided in non-Newtonian calculus, too.

Proposition 1.2 (See [2]**).** $|x \dot{\times} y|_N = |x|_N \dot{\times} |y|_N$ *for any* $x, y \in \mathbb{R}(N)$.

Proposition 1.3 (See [2]**).** *The triangle inequality with respect to non-Newtonian distance* $|\cdot|_N$, *for any* $x, y \in \mathbb{R}(N)$ *is given by*

$$|x \dot{+} y|_N \leq |x|_N \dot{+} |y|_N.$$

The non-Newtonian metric spaces provide an alternative to the metric spaces introduced in [2].

Definition 1.4 (See [2]**).** Let $X \neq \emptyset$ be a set. If a function $d_N : X \times X \to \mathbb{R}^+(N)$ satisfies the following axioms for all $x, y, z \in X$:

(NM1) $d_N(x, y) = \beta(0) = \dot{0}$ if and only if $x = y$,
(NM2) $d_N(x, y) = d_N(y, x)$,
(NM3) $d_N(x, y) \dot{\leq} d_N(x, z) \dot{+} d_N(z, y)$,

then it is called a non-Newtonian metric on X and the pair (X, d_N) is called a non-Newtonian metric space.

Proposition 1.5 (See [2]**).** *Suppose that the non-Newtonian metric* d_N *on* $\mathbb{R}(N)$ *is such that* $d_N(x, y) = |x \dot{-} y|_N$ *for all* $x, y \in \mathbb{R}(N)$, *then* $(\mathbb{R}(N), d_N)$ *is a non-Newtonian metric space.*

Definition 1.6 (See [2]**).** Let X be a vector space on $\mathbb{R}(N)$. If a function $\|\cdot\|_N : X \to \mathbb{R}^+(N)$ satisfies the following axioms for all $x, y \in X$ and $\lambda \in \mathbb{R}(N)$:

(NN1) $\|\cdot\|_N = \dot{0} \Leftrightarrow x = \dot{0}$,
(NN2) $\|\lambda \dot{\times} x\|_N = |\lambda|_N \dot{\times} \|x\|_N$,
(NN3) $\|x \dot{+} y\|_N \leq \|x\|_N \dot{+} \|y\|_N$,

then it is called a non-Newtonian norm on X and the pair $(X, \|\cdot\|_N)$ is called a non-Newtonian normed space.

Remark 1.7 (See [2]**).** Here it is easily seen that every non-Newtonian norm $\|\cdot\|_N$ on X produces a non-Newtonian metric d_N on X given by

$$d_N(x, y) = \|x \dot{-} y\|_N, \quad x, y \in X.$$

The question that inspires us is, Can the fixed point theory be applied to the non-Newtonian metric spaces? Therefore, in this paper, we give some topological properties of the relevant non-Newtonian metric space, we also introduce the concept of non-Newtonian contraction mappings and present an exact analogue of the well-known Banach contraction principle in non-Newtonian metric spaces and some theorems.

2. Some topological concepts

Now, we define some topological structures related to non-Newtonian metric spaces.

Proposition 2.1. *Let* (X, d_N) *be a non-Newtonian metric space. Then we have the following inequality:*

$$\left|d_N(x,z) \dot{-} d_N(y,z)\right|_N \dot{\leq} d_N(x,y)$$

for all $x, y, z \in X$.

Proof. By the non-Newtonian triangle inequality, it is seen that

$$d_N(x,z) \dot{\leq} d_N(x,y) \dot{+} d_N(y,z) \Longrightarrow d_N(x,z) \dot{-} d_N(y,z) \dot{\leq} d_N(x,y),$$
$$d_N(y,z) \dot{\leq} d_N(y,x) \dot{+} d_N(x,z) \Longrightarrow d_N(y,z) \dot{-} d_N(x,z) \dot{\leq} d_N(y,x).$$

Thus, by the definition of $|\cdot|_N$ and from the above inequalities, we have

$$\left|d_N(x,z) \dot{-} d_N(y,z)\right|_N \dot{\leq} d_N(x,y).$$ □

Definition 2.2. Let (X, d_N) be a non-Newtonian metric space, $x \in X$ and $\varepsilon \dot{>} \dot{0}$, we now define a set

$$B_\varepsilon^N(x) = \left\{ y \in X : d_N(x,y) \dot{<} \varepsilon \right\},$$

which is called a non-Newtonian open ball of radius ε with center x. Similarly, one describes the non-Newtonian closed ball as

$$\bar{B}_\varepsilon^N(x) = \left\{ y \in X : d_N(x,y) \dot{\leq} \varepsilon \right\}.$$

Example 2.3. Consider the non-Newtonian metric space $(\mathbb{R}^+(N), d_N^*)$. From the definition of d_N^*, we can verify that the non-Newtonian open ball of radius $\varepsilon \dot{>} \dot{1}$ with center x_0 appears as

$$\left(x_0 \dot{-} \varepsilon, x_0 \dot{+} \varepsilon\right) \subset \mathbb{R}^+(N).$$

Definition 2.4. Let (X, d_N) be a non-Newtonian metric space and $A \subset X$. Then we call $x \in A$ a non-Newtonian interior point of A if there exists an $\varepsilon \dot{>} \dot{1}$ such that $B_\varepsilon^N(x) \subset A$. The collection of all interior points of A is called the non-Newtonian interior of A and is denoted by $\mathrm{int}_N(A)$.

Definition 2.5. Let (X, d_N) be a non-Newtonian metric space and $A \subset X$. If every point of A is a non-Newtonian interior point of A, i.e., $A = \mathrm{int}_N(A)$, then A is called a non-Newtonian open set.

Lemma 2.6. *Let (X, d_N) be a non-Newtonian metric space. Each non-Newtonian open ball of X is a non-Newtonian open set.*

Proof. Let $x \in X$ and let $B_{\varepsilon}^{N}(x)$ be a non-Newtonian open ball. For $y \in B_{\varepsilon}^{N}(x)$, if we let

$$\delta = \varepsilon \dot{-} d_N(x, y) \quad \text{and} \quad z \in B_{\delta}^{N}(y),$$

then

$$d_N(y, z) \dot{<} \varepsilon \dot{-} d_N(x, y),$$

from which we conclude that

$$d_N(x, z) \dot{\leq} d_N(x, y) \dot{+} d_N(y, z) \dot{<} \varepsilon.$$

This shows that $z \in B_{\varepsilon}^{N}(x)$, which means $B_{\delta}^{N}(y) \subset B_{\varepsilon}^{N}(x)$. Thus $B_{\varepsilon}^{N}(x)$ is a non-Newtonian open set. □

Lemma 2.7. *Let (X, d_N) be a non-Newtonian metric space. Then X and $\emptyset$ are non-Newtonian open sets.*

Lemma 2.8. *The union of finite, countable or uncountable, family of non-Newtonian open sets is also a non-Newtonian open set.*

Lemma 2.9. *The intersection of any finite family of non-Newtonian open sets is also a non-Newtonian open set.*

Proof of Lemma 2.9. Let B_1 and B_2 be two non-Newtonian open sets and $y \in B_1 \cap B_2$. Then there are $\delta_1, \delta_2 \dot{>} \dot{0}$ such that

$$B_{\delta_1}^{N}(y) \subset B_1^{N} \quad \text{and} \quad B_{\delta_2}^{N}(y) \subset B_2^{N}.$$

Letting δ be the smaller of δ_1 and δ_2, we conclude that

$$B_{\delta}^{N}(y) \subset B_1^{N} \cap B_2^{N}.$$

Hence the intersection of any finite family of non-Newtonian open sets is a non-Newtonian open set. □

Theorem 2.10. *Every non-Newtonian metric space is a topological space based on the set of all non-Newtonian open sets.*

Proof. By the results obtained until now, the proof is clear. □

Definition 2.11. Let (X, d_N) be a non-Newtonian metric space. A point $x \in X$ is said to be a non-Newtonian limit point of $S \subset X$ if and only if

$$\left(B_{\varepsilon}^{N}(x) \setminus \{x\}\right) \cap S \neq \emptyset \quad \text{for every } \varepsilon \dot{>} \dot{0}.$$

The set of all non-Newtonian limit points of the set S is denoted by S'^{N}.

Definition 2.12. Let (X, d_N) be a non-Newtonian metric space. A set $S \subset X$ is said to be non-Newtonian closed in (X, d_N) if S contains all of its non-Newtonian limit points.

Proposition 2.13. *Let (X, d_N) be a non-Newtonian metric space and $S \subset X$. Then $S \cup S^{N}$ is a non-Newtonian closed set. This set is called a non-Newtonian closure of the set S, which is denoted by $\bar{S}^{N}$.*

Proposition 2.14. *Let (X, d_N) be a non-Newtonian metric space and $S \subset X$. The set S is non-Newtonian closed if and only if $X \setminus S$, the complement of S, is non-Newtonian open.*

The above propositions can be easily proven by the definition of the non-Newtonian closed set.

Definition 2.15. Let $\left(X, d_N^X\right)$ and $\left(Y, d_N^Y\right)$ be two non-Newtonian metric spaces and let $f : X \to Y$ be a function. If f satisfies the requirement that, for every $\varepsilon \dot{>} \dot{0}$, there exists $\delta \dot{>} \dot{0}$ such that

$$f\left(B_\delta^N(x)\right) \subset B_\varepsilon^N(f(x)),$$

then f is said to be non-Newtonian continuous at $x \in X$.

Example 2.16. Given a non-Newtonian metric space (X, d_N), define a non-Newtonian metric on $X \times X$ by

$$p\big((x_1, x_2), (y_1, y_2)\big) = d_N(x_1, y_1) \dot{+} d_N(x_2, y_2).$$

Then the non-Newtonian metric $d_N : X \times X \to (\mathbb{R}^+(N), |\cdot|_N)$ is non-Newtonian continuous on $X \times X$. To show this, let (y_1, y_2), $(x_1, x_2) \in X \times X$. Since we have

$$\left|d_N(y_1, y_2) \dot{-} d_N(x_1, x_2)\right|_N \dot{\leq} d_N(x_1, y_1) \dot{+} d_N(x_2, y_2),$$

it is clear that d_N is non-Newtonian continuous on $X \times X$.

Now, we emphasize some properties of convergent sequences in a non-Newtonian metric space.

Definition 2.17 (See [2]**).** A sequence (x_n) in a metric space $X = (X, d_N)$ is said to be convergent if for every given $\varepsilon \dot{>} \dot{0}$ there exist an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $x \in X$ such that $d_N(x_n, x) \dot{<} \varepsilon$ for all $n > n_0$, and it is denoted by

$$^N\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \xrightarrow{N} x, \quad \text{as } n \to \infty.$$

Definition 2.18. A sequence (x_n) in a non-Newtonian metric space $X = (X, d_N)$ is said to be non-Newtonian Cauchy if for every $\varepsilon \dot{>} \dot{0}$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$d_N(x_n, x_m) \dot{<} \varepsilon \quad \text{for all } m, n > n_0.$$

Similarly, if for every non-Newtonian open ball $B_\varepsilon^N(x)$, there exists a natural number n_0 such that $n > n_0$, $x_n \in B_\varepsilon^N(x)$, then the sequence (x_n) is said to be non-Newtonian convergent to x.

The space X is said to be non-Newtonian complete if every non-Newtonian Cauchy sequence in X converges (see [2]).

Proposition 2.19 (See [2]**).** *Let $X = (X, d_N)$ be a non-Newtonian metric space. Then*

(i) *a convergent sequence in X is bounded and its limit is unique,*
(ii) *a convergent sequence in X is a Cauchy sequence in X.*

Lemma 2.20. *Let (X, d_N) be a non-Newtonian metric space, (x_n) a sequence in X and $x \in X$. Then $x_n \xrightarrow{N} x$ $(n \to \infty)$ if and only if $d_N(x_n, x) \xrightarrow{N} \dot{0}$ $(n \to \infty)$.*

Proof. Suppose that the sequence (x_n) is non-Newtonian convergent to x. That is, for every $\varepsilon \dot{>} \dot{1}$, there is a natural number n_0 such that $d_N(x_n, x) \dot{<} \varepsilon$ whenever $n > n_0$. Thus we have the following inequality:

$$-\varepsilon \dot{<} d_N(x_n, x) \dot{<} \varepsilon \quad \text{for all } n > n_0.$$

This means

$$|d_N(x_n, x)|_N \dot{<} \varepsilon \quad \text{for all } n > n_0,$$

which implies that the sequence $d_N(x_n, x)$ is non-Newtonian convergent to $\dot{0}$. □

The converse of the above result can be easily verified.

Lemma 2.21. *Let (X, d_N) be a non-Newtonian metric space and let (x_n) be a sequence in X. If the sequence (x_n) is non-Newtonian convergent, then the non-Newtonian limit point is unique.*

Theorem 2.22. *Let $\left(X, d_N^X\right)$ and $\left(Y, d_N^Y\right)$ be two non-Newtonian metric spaces, $f : X \to Y$ a mapping and (x_n) any sequence in X. Then f is non-Newtonian continuous at the point $x \in X$ if and only if $f(x_n) \xrightarrow{N} f(x)$ for every sequence (x_n) with $x_n \xrightarrow{N} x$ $(n \to \infty)$.*

Proof. Suppose that f is non-Newtonian continuous at the point x and that $x_n \xrightarrow{N} x$. From the non-Newtonian continuity of f, we have that, for every $\varepsilon \dot{>} \dot{0}$, there exists $\delta \dot{>} \dot{0}$ such that

$$f\left(B_\delta^N(x)\right) \subset B_\varepsilon^N(f(x)).$$

Since $x_n \xrightarrow{N} x$ $(n \to \infty)$, there exists no x_n such that $n > n_0$ implies that $x_n \in B_\delta^N(x)$. By virtue of the above inclusion,

$$f(x_n) \in B_\varepsilon^N(f(x))$$

and hence

$$f(x_n) \xrightarrow{N} f(x) \quad (n \to \infty).$$

Conversely, assume that f is not non-Newtonian continuous at x. That is, there exists an $\varepsilon \dot{>} \dot{1}$ such that for each $\delta \dot{>} \dot{1}$, we have $x' \in X$ with

$$d_N^X\left(x^{'N}, x\right) \dot{<} \delta,$$

but $d_N^Y(f(x'), f(x)) \dot{\geq} \varepsilon$. Now take any sequence of real numbers (δ_n) such that $\delta_n \xrightarrow{N} \dot{0}$ and $\delta_n \dot{>} \dot{0}$ for each n. For each n, select n' that satisfies the above inequality and call this x'_n. It is clear that $x'_n \xrightarrow{N} x$, but $f(x'_n)$ is not non-Newtonian convergent to $f(x)$. Hence we see that if f is not non-Newtonian continuous, then not every sequence (x_n) with $x_n \xrightarrow{N} x$ will yield a sequence

$$f(x_n) \xrightarrow{N} f(x).$$

Taking the contrapositive of this statement demonstrates that the condition of the theorem is sufficient. □

Theorem 2.23. *Let (X, d_N) be a non-Newtonian metric space and $S \subset X$. Then*

- (i) *a point $x \in X$ belongs to $\bar{S}$ if and only if there exists a sequence (x_n) in S such that $x_n \xrightarrow{N} x$ $(n \to \infty)$,*
- (ii) *the set S is non-Newtonian closed if and only if every non-Newtonian convergent sequence in S has a non-Newtonian limit point that belongs to S.*

Proof. (i) Let $x \in S$. If we consider a sequence (x_n) with $x_n = x$ for all n, then this is a sequence in S such that $x_n \xrightarrow{N} x$. Let $x \in S'_N$ Thus, for $\varepsilon_n = \dot{1} \dot{+} \dot{1} \mathbin{\dot{/}} \dot{n}$, we have $B^N_{\varepsilon_n}(x) \cap S \neq \emptyset$. By choosing $x_n \in B^N_{\varepsilon_n}(x) \cap S$, we can set a sequence (x_n) in S such that $x_n \xrightarrow{N} x$ $(n \to \infty)$. It is easy to prove the converse.

(ii) This is easily seen from (i). □

3. Main results

We first define the fixed point theorem on non-Newtonian metric spaces and give some examples.

Definition 3.1. Let X be a set and T a map from X to X. A fixed point of T is a point $x \in X$ such that $Tx = x$. In other words, a fixed point of T is a solution of the functional equation $Tx = x$, $x \in X$.

Definition 3.2. Suppose that (X, d_N) is a non-Newtonian complete metric space and $T : X \to X$ is any mapping. The mapping T is said to satisfy a non-Newtonian Lipschitz condition with $k \in \mathbb{R}(N)$ if

$$d\big(T(x), T(y)\big) \mathrel{\dot{\leq}} k \mathbin{\dot{\times}} d(x, y)$$

holds for all $x, y \in X$. If $k \mathrel{\dot{<}} \dot{1}$, then T is called a non-Newtonian contraction mapping. We now state and prove the non-Newtonian contraction mapping theorem or the non-Newtonian Banach fixed point theorem.

Theorem 3.3. *Let T be a non-Newtonian contraction mapping on a non-Newtonian complete metric space X. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. Define the "iterative sequence" $\{x_n\}$,

$$x_0,\ x_1 = Tx_0,\ x_2 = Tx_1 = T^2 x_0, \ldots, x_n = T^n x_0,$$

and now we show that the sequence (x_n) is a non-Newtonian Cauchy sequence:

$$\begin{aligned} d_N(x_m, x_n) &= d_N\left(T^m x_0, T^n x_0\right) \\ &= d_N\left(T^m x_0, T^m T^{m-n} x_0\right) \\ &\dot{\le} k^{m_N} \dot{\times} d_N\left(x_0, T^{n-m} x_0\right) \\ &= k^{m_N} \dot{\times} d_N(x_0, x_{n-m}) \\ &\dot{\le} k^{m_N} \dot{\times} \left\{d_N(x_0, x_1) \dot{+} \cdots \dot{+} d_N(x_{n-m-1}, x_{n-m})\right\} \\ &\dot{\le} k^{m_N} \dot{\times} d_N(x_0, x_1) \left\{\dot{1} \dot{+} k \dot{+} k^{2_N} \dot{+} \cdots \dot{+} k^{n-m-1_N}\right\} \\ &\dot{\le} \frac{k^{m_N} \dot{\times} d_N(x_0, x_1)}{\dot{1} \dot{-} k}. \end{aligned}$$

If $m < n$, then since $k^{m_N} \dot{<} \dot{1}$ and $d_N(x_0, x_1) \in (N)$ is fixed, we can make

$$\frac{k^{m_N} \dot{\times} d_N(x_0, x_1)}{\dot{1} \dot{-} k}$$

as small as we want by taking m sufficiently large. This shows that $\{x_n\}$ is a non-Newtonian Cauchy sequence. Since (X, d_N) is non-Newtonian complete, there exists a point x in X such that $x_n \xrightarrow{N} x$. We now show that x is a fixed point of the mapping T. It follows from the triangle inequality that

$$\begin{aligned} d_N(x, Tx) &\dot{\le} d_N(x, x_n) \dot{+} d_N(x_n, Tx) \\ &\dot{\le} d_N(x, x_n) \dot{+} k \dot{\times} d_N(x_{n-1}, x). \end{aligned}$$

Since $x_n \xrightarrow{N} x$, it follows that $d_N(x, Tx) = \dot{0}$ and hence $Tx = x$. This shows that x is a fixed point of T. We conclude the proof by showing that x is the only fixed point. Suppose that x_1 is also a fixed point, that is, suppose $Tx_1 = x_1$, then

$$d_N(x, x_1) = d_N(Tx, Tx_1) \dot{\le} k \dot{\times} d_N(x, x_1)$$

and since $k \dot{<} \dot{1}$, this implies that $d_N(x, x_1) = \dot{0}$ and hence $x = x_1$. □

Corollary 3.4. *Under the assumption of the above theorem the iterative sequence with arbitrary $x_0 \in X$ converges to the unique fixed point x of T. The prior error estimates are*

$$d_N(x_m, x) \dot{\le} \frac{k^{m_N}}{\dot{1} \dot{-} k} \dot{\times} d_N(x_0, x_1)$$

and the post error estimates are

$$d_N(x_m, x) \dot{\le} \frac{k}{\dot{1} \dot{-} k} \dot{\times} d_N(x_{m-1}, x_m).$$

Theorem 3.5. *Let T be a mapping on a non-Newtonian complete metric space X into itself. Let T be a non-Newtonian contraction on a closed ball*

$$\bar{B}^N_{\dot{r}}\left(x_0\right) = \left\{x \in X : d_N\left(x, x_0\right) \dot{\le} \dot{r}\right\}.$$

Suppose that $d(x_0, Tx_0) \mathrel{\dot{<}} (\dot{1} \mathbin{\dot{-}} k)\,\dot{r}$. *Then the iterative sequence defined by*

$$x_n = T^n x_0 = T x_{n-1}$$

converges to an $x \in \bar{B}^N_{\dot{r}}(x_0)$ *and this* x *is the unique fixed point of* T.

Proof. Setting $m = 0$ in

$$d_N(x_m, x_n) \mathrel{\dot{\le}} \frac{k^m \mathbin{\dot{\times}} d_N(x_0, x_1)}{\dot{1} \mathbin{\dot{-}} k},$$

we have

$$d_N(x_0, x_n) \mathrel{\dot{\le}} \frac{d_N(x_0, x_1)}{\dot{1} \mathbin{\dot{-}} k}.$$

It follows from $d_N(x_0, Tx_0) \mathrel{\dot{<}} (\dot{1} \mathbin{\dot{-}} k)\,\dot{r}$ that

$$d_N(x_0, x_n) \mathrel{\dot{\le}} \frac{d_N(x_0, x_1)}{\dot{1} \mathbin{\dot{-}} k} \mathrel{\dot{<}} \dot{r}.$$

This shows that all the x_n's are in $\bar{B}^N_{\dot{r}}(x_0)$. Since $x_n \to x$ and $\bar{B}^N_{\dot{r}}(x_0)$ is closed, it follows that

$$x \in \bar{B}^N_{\dot{r}}(x_0).$$

The assertion of the theorem now follows from the proof of the non-Newtonian contraction mapping theorem. ☐

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References

[1] A. E. Bashirov, E. M. Kurpınar and A. Özyapıcı, *Multiplicative calculus and its applications.* J. Math. Anal. Appl. **337** (2008), 36–48.

[2] A. F. Çakmak and F. Başar, *Some new results on sequence spaces with respect to non-Newtonian calculus.* J. Inequal. Appl. **2012** (2012), doi:10.1186/1029-242X-2012-228, 17 pages.

[3] B. Choudhary and S. Nanda, *Functional Analysis with Applications.* John Wiley and Sons, New York, 1990.

[4] M. Grossman and R. Katz, *Non-Newtonian Calculus.* Lowell Technological Institute, 1972.

[5] X. He, M. Song and D. Chen, *Common fixed points for weak commutative mappings on a multiplicative metric space.* Fixed Point Theory Appl. **2014** (2014), doi:10.1186/1687-1812-2014-48, 9 pages.

[6] M. Özavşar and A. C. Çevikel, *Fixed points of multiplicative contraction mappings on multiplicative metric spaces.* Preprint, arXiv:1205.5131v1 [matn.GN], 2012.

[7] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications.* Yokohama Publishers, Yokohama, 2000.

[8] A. Uzer, *Multiplicative type complex calculus as an alternative to the classical calculus.* Comput. Math. Appl. **60** (2010), 2725–2737.

Demet Binbaşıoğlu
Department of Mathematics
Faculty of Arts and Sciences
Gaziosmanpaşa University
Tokat-60240
Turkey
e-mail: `demetbinbasi@hotmail.com`

Serkan Demiriz
Department of Mathematics
Faculty of Arts and Sciences
Gaziosmanpaşa University
Tokat-60240
Turkey
e-mail: `serkandemiriz@gmail.com`

Duran Türkoğlu
Department of Mathematics
Faculty of Sciences
Gazi University
06500-Teknikokullar, Ankara
Turkey
e-mail: `dturkoglu@gazi.edu.tr`