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Some Common Fixed Point Theorems for Self Mappings in Vector Metric Spaces

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Abstract. In this paper, we prove some theorems about common fixed point for two self mappings satisfying some general contractive conditions in vector metric spaces. Presented results are generalizations of the some well-known recent fixed point theorems.

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1. Introduction

A vector metric space is generalization of metric space. This metric is Riesz space valued. Actually, both vector metric and cone metric are vector space valued. One of the differences between definition of vector metric and definition of cone metric is that there exists a cone due to the natural existence of ordering on Riesz space. The other difference is that our definition eliminates the requirement for the vector space to have a topological structure.

Recently, many authors have studied on common fixed point theorems for weakly compatible pairs ([2],[3],[7]). Some of these works is gived in cone metric spaces ([1],[6],[7],[8]). Çevik and Altun ([4],[5]) prove Baire theorem and Banach fixed point theorem on vector spaces and give some theorems on point of coincidence and common fixed points for two self mappings satisfying some general contractive conditions in vector spaces.

Let T and S be self maps of a set X. If $y = Tx = Sx$ for some $x \in X$, then y is said to be a point of coincidence and x is said to be a coincidence point of T and S. If T and S are weakly compatible, that is, they are commuting at their coincidence point on X, then the point of coincidence y is the unique common fixed point of these maps [1].

As definition; a Riesz space is an ordered vector space and a lattice. Let E be a Riesz space with the positive cone $E_+ = \{x \in E : x \geq 0\}$. If (a_n) is a decreasing sequence in E such that $\inf a_n = a$, we write $a_n \downarrow a$.

2. Preliminaries

We shall require the following definitions in the sequel.

Definition 1 ([5])**.** *The Riesz space E is said to be Archimedean if $\frac{1}{n}a \downarrow 0$ holds for every $a \in E_+$.*

Definition 2 ([5])**.** *A sequence (b_n) is said to order convergent (or $\circ-$convergent) to b if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $\mid b_n - b \mid \leq a_n$ for all n and written $b_n \xrightarrow{\circ} b$ or $\circ - \lim b_n = b$, where $\mid a \mid = \sup\{a, -a\}$ for any $a \in E$.*

Definition 3 ([5])**.** *A sequence (b_n) is said to be order-Cauchy (or $\circ-$Cauchy) if there exists a sequence (a_n) in such that $a_n \downarrow 0$ and $\mid b_n - b_{n+p} \mid \leq a_n$ holds for all n and p.*

Definition 4 ([5])**.** *The Riesz space E is said to be $\circ - Cauchy$ complete if every $\circ - Cauchy$ sequence is $\circ - convergent$.*

Definition 5 ([5])**.** *Let X be a non-empty set and E be a Riesz space. The function $d : X \times X \to E$ is said to be a vector metric (or $E-$metric) if it is satisfying the following properties:*

(i) $d(x, y) = 0$ *if and only if* $x = y$,

(ii) $d(x, y) \leq d(x, z) + d(y, z)$

for all x, y, $z \in X$. Also the triple (X, d, E) is said to be vector metric space.

For arbitrary elements x, y, z, w of a vector metric space, the following statements are satisfied.

a. $0 \leq d(x,y)$;
b. $d(x,y) = d(y,x)$;
c. $\mid d(x,z) - d(y,z) \mid \leq d(x,y)$;
d. $\mid d(x,z) - d(y,w) \mid \leq d(x,y) + d(z,w)$.

Definition 6 ([5]). *A sequence (x_n) in a vector metric space (X,d,E) vectorial converges (or $E-$converges) to some $x \in E$, written $x_n \overset{d,E}{\rightarrow} x$, if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ for all n.*

Definition 7 ([5]). *A sequence (x_n) is called $E-$ Cauchy sequence whenever there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$ holds for all n and p.*

Remark 1 ([5]). *A vector metric space X is called $E-$complete if each $E-$Cauchy sequence in X, $E-$converges to a limit in X.*

There are the following properties;

If $x_n \overset{d,E}{\rightarrow} x$, then

$(i^{'})$ The limit x is unique,

$(ii^{'})$ Every subsequence of (x_n) $E-$converges to x,

$(iii^{'})$ If also $y_n \overset{d,E}{\rightarrow} y$, then $d(x_n, y_n) \overset{\circ}{\rightarrow} d(x,y)$.

When $E = \mathbb{R}$, the concepts of vectorial convergence and convergence in metric are the same. When also $X = E$ and d is the concepts of absolute valued vector metric, vectorial convergence and convergence in order are the same. When $E = \mathbb{R}$, the concepts of E-Cauchy sequence and Cauchy sequence are the same.

Remark 2 ([5]). *It is well known that $\mathbb{R}^2$ is a Riesz space with coordinatwise ordering defined by*

$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2$ and $y_1 \leq y_2$ for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Again $\mathbb{R}^2$ is a Riesz space with lexicographical ordering defined by $(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 < x_2$ or $x_1 = x_2$, $y_1 \leq y_2$. Note that $\mathbb{R}^2$ is Archimedean with coordinatwise ordering but not with lexicographical ordering.

Remark 3 ([5]). *If E is a Riesz space and $a \leq ka$ where $a \in E_+$, $k \in [0,1)$, then $a = 0$.*

3. Main Results

Theorem 1. *Let X be a vector metric space with E is Archimedean. Suppose S a continuous self map on X and T be any self map on X that commutes with S. Suppose the following conditions is satisfied;*

i) $T(X) \subseteq S(X)$

ii) *for all* $x,\, y \in X$, $d(Tx,Ty) \le ku(x,y)$ *where* $k \in (0, \frac{1}{2})$ *is a constant and* $u(x,y) \in \{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\}$

iii) $S(X)$ *or* $T(X)$ *is* $E-$*complete subspace of* X.

Then S *and* T *have the unique common fixed point.*

Proof. From the condition (i) implies that starting with an arbitrary $x_0 \in X$, so we can construct a sequence $\{y_n\}$ of points in X such that $y_n = Tx_n = Sx_{n+1}$, for all $n \ge 0$.

Our aim is prove that $\{y_n\}$ is an $E-Cauchy$ sequence.

Firstly, we show that

$$d(y_n, y_{n+1}) \le \frac{k}{1-k} d(y_{n-1}, y_n) \text{ for all } n \ge 1. \; ...(1)$$

In real,

$$d(y_n, y_{n+1}) = d(Tx_n, Tx_{n+1}) \le ku_n \; . \; ...(2)$$

$$\begin{aligned} \text{where } u_n \in \{ & d(Sx_n, Sx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}), d(Sx_n, Tx_{n+1}), \\ & d(Sx_{n+1}, Tx_n)\} \\ = \{ & d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), d(y_n, y_n)\} \\ = \{ & d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), 0\}. \end{aligned}$$

From (2) it follows four cases;

(I) $d(y_n, y_{n+1}) \le kd(y_{n-1}, y_n) \le \frac{k}{1-k} d(y_{n-1}, y_n)$

(II) $d(y_n, y_{n+1}) \le kd(y_n, y_{n+1})$ and so $d(y_n, y_{n+1}) = 0$. In this case (1) follows immediately, because $k < \frac{k}{1-k}$.

(III) $d(y_n, y_{n+1}) \le kd(y_{n-1}, y_{n+1}) \le kd(y_{n-1}, y_n) + kd(y_n, y_{n+1})$. It follows that (1) holds.

(IV) $d(y_n, y_{n+1}) \le k.0 = 0$ and so $d(y_n, y_{n+1}) = 0$. Hence (1) holds.

Thus by putting $\lambda = \frac{k}{1-k}$, $d(y_n, y_{n+1}) \le \lambda d(y_{n-1}, y_n)$. Now, by using (1) we have

$d(y_n, y_{n+1}) \le \lambda d(y_{n-1}, y_n) \le ... \le \lambda^n d(y_0, y_1)$, for all $n \ge 1$.

Now, for $n > m$ we have

$$\begin{aligned} d(y_n, y_m) &\le d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + ... + d(y_{m+1}, y_m) \\ &\le \left(\lambda^{n-1} + \lambda^{n-2} + ... + \lambda^m\right) d(y_0, y_1) \\ &\le \frac{\lambda^m}{1-\lambda} d(y_0, y_1). \end{aligned}$$

Now, since E is Archimedean then (y_n) is an $E-Cauchy$ sequence. Since the range of S contains the range of T and the range of at least one is $E-$ complete, there exists a $z \in S(X)$ such that $Sx_n \xrightarrow{d,E} z$. Hence there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(Sx_n, z) \le a_n$.

Therefore, $y_n = Tx_n = Sx_{n+1} \xrightarrow{d,E} z$.

Now, we show that $Sz = Tz = z$. In this way, note that

$d(Sz,Tz) \leq d(Sz,TSx_n) + d(TSx_n,Tz)$, for all $n \geq 1$.

Also we have $d(TSx_n,Tz) \leq ku_n$ for all $n \geq 1$, where

$u_n \in \{d(S^2x_n,Sz), d(S^2x_n,TSx_n), d(Sz,Tz), d(S^2x_n,Tz), d(Sz,TSx_n)\}$.

Choose a natural number n_0 such that for all $n \geq n_0$, because

$TSx_n = STx_n \overset{d,E}{\to} Sz$ and $S^2x_n \overset{d,E}{\to} Sz$, then there exists sequences (a_n) and (b_n) in E such that $a_n \downarrow 0$ and $b_n \downarrow 0$ and we have $d(Sz,TSx_n) \leq a_n$ and $d(S^2x_n,Sz) \leq b_n$. Thus we obtain the following cases;

Case 1: $d(Sz,Tz) \leq d(Sz,TSx_n) + kd(S^2x_n,Sz) \leq a_n + kb_n$

Case 2: $d(Sz,Tz) \leq d(Sz,TSx_n) + kd(S^2x_n,TSx_n)$
$\leq d(Sz,TSx_n) + k(d(S^2x_n,Sz) + d(Sz,TSx_n))$
$\leq a_n + k(b_n + a_n)$

Case 3: $d(Sz,Tz) \leq d(Sz,TSx_n) + kd(Sz,Tz) \leq \frac{a_n}{1-k}$

Case 4: $d(Sz,Tz) \leq d(Sz,TSx_n) + kd(S^2x_n,Tz)$
$\leq d(Sz,TSx_n) + k(d(S^2x_n,Sz) + d(Sz,Tz))$
$\leq \frac{a_n+kb_n}{1-k}$

Case 5: $d(Sz,Tz) \leq d(Sz,TSx_n) + kd(Sz,STx_n)$
$\leq (1+k)\,d(Sz,TSx_n)$
$\leq (1+k)a_n$
$\leq 2a_n$.

Since the infimum of sequences on the right side of last inequality are zero, then $d(Sz,Tz) = 0$ i.e. $Sz = Tz$. ...(3)

Thus,

$d(Sz,z) \leq d(Tz,Tx_n) + d(Tx_n,z) \leq d(Tx_n,z) + kv_n$,

where $v_n \in \{d(Sx_n,Sz), d(Sx_n,Tx_n), d(Sz,Tz), d(Sx_n,Tz), d(Sz,Tx_n)\}$
$= \{d(Sx_n,Sz), d(Sx_n,Tx_n), 0, d(Sz,Tx_n)\}$.

Choose a natural number n_0 such that for all $n \geq n_0$ we have

$d(Sx_n,z) \leq c_n$ and $d(Tx_n,z) \leq d_n$, as $c_n \downarrow 0$ and $d_n \downarrow 0$.

Again, we have the following cases;

Case 1': $d(Sz,z) \leq d(Tx_n,z) + kd(Sx_n,Sz)$
$\leq d(Tx_n,z) + kd(Sx_n,z) + kd(z,Sz)$
$\leq \frac{d_n+kc_n}{1-k}$

Case 2': $d(Sz,z) \leq d(Tx_n,z) + kd(Sx_n,Tx_n)$
$\leq d(Tx_n,z) + kd(Sx_n,z) + kd(z,Tx_n)$
$\leq d_n + kc_n + kd_n$
$= (1+k)\,d_n + kc_n$

Case 3': $d(Sz,z) \leq d(Tx_n,z) + k.0 = d(Tx_n,z) \leq d_n$

Case 4': $d(Sz,z) \leq d(Tx_n,z) + kd(Sz,Tx_n)$
$\leq d(Tx_n,z) + kd(Sz,z) + kd(z,Tx_n)$
$\leq d_n + kc_n + kd_n$

$= (1+k)\, d_n + kc_n.$

Therefore, since the infimum of sequences on the right side of last inequality are zero, then $d(Sz, z)$ that is $Sz = z$.

Finally from (3) $Sz = Tz = z$ and so z is a common fixed point for S and T. If z_1 is another point of common then $z_1 = Tz_1 = Sz_1$. Now from (2)

$d(z, z_1) = d(Tz, Tz_1) \leq ku(z, z_1)$

where

$u(z, z_1) \in \{d(Sz, Sz_1), d(Sz, Tz), d(Sz_1, Tz_1), d(Sz, Tz_1), d(Sz_1, Tz)\}$
$= \{0, d(z, z_1)\}.$

Hence $d(z, z_1) = 0$ that is $z = z_1$.

Therefore z is an unique common fixed point for S and T. □

Corollary 1. *Let X be an vector metric space with E is Archimedean. Suppose S is a continuous self map on X and T be any self map on X that commutes with S. Further let S and T satisfy $T(X) \subseteq S(X)$ and that for some constant $k \in (0,1)$ and every $x, y \in X$, $d(Tx, Ty) \leq kd(Sx, Sy)$. Then S and T have an unique common fixed point.*

Theorem 2. *Let X be a vector metric space with E is Archimedean. Suppose S^2 is a continuous self map on X and T be any self map on X that commutes with S. Suppose the following conditions is satisfied;*

i) $TS(X) \subseteq S^2(X)$
ii) *for all x, $y \in X$, $d(Tx, Ty) \leq ku(x, y)$ where $k \in (0, \frac{1}{2})$ is a constant and* $u(x, y) \in \{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}$
iii) *$S(X)$ or $T(X)$ is $E-$complete subspace of X.*

Then S and T have the unique common fixed point.

Proof. From the condition (i) implies that starting with an arbitrary $x_0 \in SX$, we can construct a sequence $\{y_n\}$ of points in SX such that $y_n = Tx_n = Sx_{n+1}$, for all $n \geq 0$.

Now $Sy_{n+1} = STx_{n+1} = TSx_{n+1} = Ty_n = z_n$, $n \geq 1$. We prove that $\{z_n\}$ is an $E - Cauchy$ sequence and hence convergent to some $z \in X$ (as in the Theorem (1)). Further we shall show that $S^2 z = TSz$.

Because; $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} STx_n = \lim_{n\to\infty} TSx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} z_n = z$, it follows that

$\lim_{n\to\infty} S^4 x_n = \lim_{n\to\infty} S^3 Tx_n = \lim_{n\to\infty} TS^3 x_n = S^2 z$, since S^2 is continuous. Now, we obtain

$d(S^2 z, TSz) \leq d(S^2 z, S^3 Tx_n) + d(S^3 Tx_n, TSz) \leq d(S^2 z, S^3 Tx_n) + ku_n,$

where

$u_n \in \{d(S^4 x_n, S^2 z), d(S^4 x_n, TS^3 x_n), d(S^2 z, TSz),$
$d(S^4 x_n, TSz), d(S^2 z, TS^3 x_n)\}.$

Choose a natural number n_0 such that for all $n \geq n_0$, because $S^3Tx_n \overset{d,E}{\to} S^2z$ and $S^4x_n \overset{d,E}{\to} S^2z$, then we have $d(S^2z, S^3Tx_n) \leq a_n$ and $d(S^4x_n, S^2z) \leq b_n$, as $a_n \downarrow 0$ and $b_n \downarrow 0$. Again we have the following cases;

Case 1: $d(S^2z, TSz) \leq d(S^2z, S^3Tx_n) + kd(S^4x_n, S^2z) \leq a_n + kb_n$

Case 2: $d(S^2z, TSz) \leq d(S^2z, S^3Tx_n) + kd(S^4x_n, TS^3z)$
$\leq d(S^2z, S^3Tx_n) + k(d(S^4x_n, S^2z) + kd(S^2z, TS^3x_n))$
$\leq a_n + k(b_n + a_n) = (1+k)a_n + kb_n$.

Case 3: $d(S^2z, TSz) \leq d(S^2z, S^3Tx_n) + kd(S^2z, TSz) \leq \frac{a_n}{1-k}$

Case 4: $d(S^2z, TSz) \leq d(S^2z, S^3Tx_n) + kd(S^4x_n, TSz)$
$\leq d(S^2z, S^3Tx_n) + k(d(S^4x_n, S^2z) + d(S^2z, TSz))$
$\leq \frac{a_n + kb_n}{1-k}$

Case 5: $d(S^2z, TSz) \leq d(S^2z, S^3Tx_n) + kd(S^2z, TS^3x_n)$
$\leq a_n + ka_n$
$\leq (1+k)a_n$.

Therefore, since the infimum of sequences on the right side of last inequality are zero, then $d(S^2z, TSz) = 0$ that is $S^2z = TSz$ and so TSz is a common fixed point for S and T. Indeed, putting in $d(Tx, Ty) \leq ku(x, y)$, $x = TSz$, $y = Sz$ we get $T(TSz) = TSz$. Because $S^2z = TSz$ i.e. $S(Sz) = T(Sz)$, we have $S(TSz) = TS^2z = T(TSz) = TSz$. □

Example 1. *Let $E = \mathbb{R}^2$ with coordinatwise ordering (since $\mathbb{R}^2$ is not Archimedean with Lexicographical ordering, then we can not use this ordering) $X = \mathbb{R}$,*
$d(x, y) = (\alpha \mid x - y \mid, \beta \mid x - y \mid)$, $\alpha, \beta > 0$, $Tx = x^2 + 2$, $Sx = 3x^2$. *Then, for all $x, y \in X$ we have*
$d(Tx, Ty) = (\alpha \mid x^2 - y^2 \mid, \beta \mid x^2 - y^2 \mid) = \frac{1}{3}d(Sx, Sy) \leq kd(Sx, Sy)$
for $k \in [\frac{1}{3}, 1)$, $T(X) = [2, \infty) \subseteq [0, \infty) = S(X)$ and $T(X)$ is $E-$complete subspace of X and S is a continuous self map on X. Therefore all conditions of Corollary (1) are satisfied. Thus T and S have an unique common fixed point.

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